Constructive uniqueness proofs of stationary vacuum Black Hole spacetimes including the case of degenerate horizons

Diplomarbeit

eingereicht von Stefan Pauliuk*,
geboren am 1. März 1982 in Herzberg(Elster)

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Tag der Verleihung des Diploms:

Das Logo der FSU entstammt der Internetseite `http://www.uni-jena.de/layout.html`

Zusammenfassung

Schwarze Löcher hoher Symmetrie sind mögliche Endprodukte eines Sternenkollapses und werden im Rahmen der Allgemeinen Relativitätstheorie (ART) beschrieben. Die Ernstgleichung ist die wesentliche Feldgleichung der ART im Falle einer stationären und axialsymmetrischen Vakuumraumzeit.
Die Grundlage der vorliegenden Arbeit bildet das von Gernot Neugebauer entwikkelte Lineare Problem für die Ernstgleichung. Mit dessen Hilfe können die Einsteingleichungen für eine stationäre und axialsymmetrische Vakuumraumzeit als Randwertproblem formuliert und für einfache Randbedingungen analytisch gelöst werden. Ein konstruktiver Eindeutigkeitsbeweis für das nicht entartete, stationär rotierende Schwarze Loch (Kerr-Metrik) konnte auf diese Weise von den beiden Autoren Reinhard Meinel und Gernot Neugebauer gegeben werden. Ziel dieser Arbeit ist es, den vorhandenen Eindeutigkeitsbeweis so abzuändern, daß er auf einen entarteten Horizont angewendet werden kann. In diesem Fall erhält man die extreme Kerrlösung. Um dieses Resultat in die Reihe der existierenden Existenz- und Eindeutigkeitsbeweise für stationäre und asymptotisch flache Vakuumlösungen mit Schwarzen Löchern einzordnen, sind weitere Schritte nötig. Diese werden diskutiert und erste Resultate werden bewiesen.
Neben der Darstellung der Resultate werden sowohl das Lineare Problem als auch Symmetrien von Raumzeiten und Schwarze Löcher ausführlich diskutiert.
Die Arbeit ist in englischer Sprache angefertigt.

Abstract

Black Holes are considerd to be possible final states of massive stellar objects and are described in the context of General Relativity. In the case of a stationary and axisymmetric vacuum spacetime, the field equations essentially reduce to the Ernst equation.
Starting from the Linear Problem for the Ernst equation, which was developed by Gernot Neugebauer, a boundary value problem can be formulated for the Ernst equation and in simple cases analytical solutions can be obtained. In this way a constructive uniqueness proof for the non-degenerate stationarily rotating Black Hole (the Kerr solution) could be obtained by Reinhard Meinel and Gernot Neugebauer. In this thesis the proof shall be extended to the case of a single degenerate horizon, which will lead to the extreme Kerr Black Hole. Further propositions are necessary to turn this statement into a uniqueness proof for a class of stationary vacuum Black Holes. These are discussed and a first step is proved.
Apart from the presentation of the results, the Linear Problem will be discussed extensively as well as spacetimes with symmetries and Black Holes in general.
This thesis is written in English.

Contents

Introduction

More than 90 years ago, Albert Einstein published his celebrated theory of General Relativity, a work of genius which still today is very demanding to anyone. It relates the geometrical properties of the world to its energy and matter content and includes Special Relativity, which he had developed ten years earlier.

It gives a deeper meaning to the principle of equivalence which arises in Newton's theory of gravity and it was inspired by Mach's ideas on relative motion. Although the theory is physically motivated, it requires a very demanding mathematical framework, the calculus of Lorentzian manifolds. These objects are the models of spacetime which emerge from the theory. They are curved four-dimensional manifolds and the metric on them allows the distance between two different points (events) to be zero, which means that a light-ray can travel between them in zero proper time.

From his theory, Einstein successfully derived a formula for Mercury's excess precession, a discrepancy which could not be explained by the classical gravitational perturbation theory. His theory also predicts an angle of deflection of light which passes the sun which is two times bigger than the value that is derived in the semi-classical photon picture.

Due to its predictions that cannot be described with common terms from everyday life, it is still controversially discussed from all walks of life and of course state-of-the-art experiments have been conducted at all times to seek for empirical justification.

The detection of gravitational waves by the inteferometers GEO600 or LISA (planned) and the gyroscope satellite 'Gravity Probe B' are examples of current research on General Relativity. Many applications of effects from General Relativity can be found in the Global Positioning System (GPS). The clocks of 24 earth-orbiting satellites have to be synchronized very precisely and for example time delation effects due to the mass of the earth must be taken into account.

There are many other facets of this theory, both theoretical and practical ones. As examples we mention the prediction of gravitational waves, the possibility to describe the universe as a whole (relativistic cosmology) and the objects that fire our imagination: Black Holes.

A Black Hole is a region in spacetime with no escape. No information or matter can leave the Black Hole region. They exist in the theory of General Relativity and some very massive dark objects in the central regions of some galaxies, including our own, are expected by the scientific community to be *supermassive Black Holes*.

This thesis is about a special class of spacetimes, namely those containing a single Black Hole and which are stationary, that is invariant unter time translation. If

symmetries are present, it is considerably easier to handle Einstein's equations.
Stationary isolated Black Holes are believed to be possible final states of stars with a mass of more than around eight times the mass of the sun at the beginning of their collapse. In particular there is a considerable interest in objects with non--vanishing angular momentum as it is rather unlikely that a star will loose all its angular momentum during its collapse. The star will shrink from an object that is far bigger that its Schwarzschild radius to a Black Hole.
They have a fascinating property, which is known as the 'no-hair-theorem': The metric of stationary vacuum Black Hole spacetimes is completely determined by three parameters: the total *mass*, the total *angular momentum* and the net *electric charge*. Without quantum mechanical theories there is no other information obtainable. However its proof is not yet complete, which is the motivation for this thesis.
The 'no-hair-theorem' is a consequence of the *Black Hole uniqueness theorems*. They name all stationary Black Hole solutions of Einstein's equations.

"The main task of the uniqueness program is to show that the static electrovac Black Hole space-times are described by the Reissner-Nordström metric, while the circular ones are represented by the Kerr-Newman metric." (Markus Heusler)

Till this day there are serious gaps in the proofs of the uniqueness theorems and this thesis will help to fill some of them.

A lot of mathematical tools have been developed to handle stationarily rotating Black Holes. In the case of stationarity and axisymmetry one can introduce a special coordinate system, the Weyl coordinates. Einstein's equation simplify remarkably in the presence of two symmetries. Indeed there is only one second order partial differential equation for a complex function remaining, the Ernst equation. It inherits the most intriguing property of Einstein's equations, the nonlinearity. Different solutions of nonlinear equations cannot be superposed, which has made it impossible to date to give a general theory of their solutions.

A typical example for a nonlinear system is the Korteweg de-Vries equation. It was derived in the 1890's from the Navier-Stokes equations of hydrodynamics to model solitons in flat water.
In the fifties a very remarkable link was found which relates the one-dimensional time dependent Schrödinger equation with an external potential to several nonlinear equations.
It was found that if the time dependence of the potential is given by the Korteweg de-Vries (KdV) equation, then there is a rather simple analytic solution for the time development of the scattering data of the wave function. Thus one can solve the Cauchy problem of the KdV-equation by solving three linear differential equations: the calculation of the Schrödinger scattering data from the potential at time zero, their time development and the 'inverse scattering method', that is the calculation of the potential out of the scattering data at a later time. A large number of similar methods have been developed in the past fifty years to obtain solutions to nonlinear equations by using Linear Problems. These are linear matrix-valued differential equations which depend on a further variable, the spectral parameter,

and which contain several unspecified functions. Their integrability condition, that is the commutation of the second order partial derivative operators, is a power series in the spectral parameter. The coefficients of the series, which are differential equations for the unspecified functions, may be nonlinear. Each nonlinear equation is related to a special choice of a *part* of these functions. The rest of them enter into the Linear Problem as ‘coefficients’. If one is able to solve the Linear Problem, one can then determine the coefficients and thus one obtains a solution of the nonlinear equations. Because of the above examples they are usually called ’Inverse (Scattering) Methods’ or ’Linear Problems’.

This short discourse briefly portrays the background of the solution technique for the Ernst equation which will be used in this thesis. There is a *linear* matrix differential equation whose solutions can be related to the ones of the Ernst equation. In the particular form which is used in this thesis it was found by Gernot Neugebauer. It is possible to impose boundary conditions and to derive solutions which respect them. We will make use of this concept when we solve the Linear Problem for a single Black Hole.

This procedure allows one to give *constructive uniqueness proofs* for certain classes of spacetimes. Reinhard Meinel and Gernot Neugebauer were able to construct the family of non-extreme Kerr Black Holes uniquely from boundary conditions. In this thesis this proof will be extended to the extreme Kerr Black Hole. Together with the Rigidity theorem which roughly states that stationary Black Holes are either static or axisymmetric, the above mentioned authors have found a uniqueness proof for the family of stationary and axisymmetric vacuum Black Holes with non-degenerate horizon. It must be mentioned that the proof of the Rigidity theorem in the version that has to be applied here is not considered complete for spacetimes with ergoregions such as the Kerr Black Hole.

The results of this thesis together with a not yet existing version of the Rigidity theorems that includes degenerate horizons will lead to the extension of the uniqueness proof of stationary vacuum Black Holes to the case of a degenerate horizon.
The extension of the existing proof is technical and its main purpose is to fill a gap, as in the available proofs degenerate horizons are excluded.
We hope that the comprehensive general considerations will help the reader towards a better understanding of General Relativity and the mathematics behind it and further we are of the opinion that the detailed description of the proof of Meinel and Neugebauer and its extension will be useful to appreciate the concept of these powerful methods.

Now we turn to the mathematical aspects of General Relativity. The concepts of absolute time and absolute space are replaced by a four-dimensional spacetime with a causal structure. In order to make reasonable statements in this framework one has to find precise mathematical notions of physical concepts. This way one can find one's way in this strange world which is supposed to model the gravitational field on large scales.

A *spacetime* $(\mathcal{M}, g)$ is a 3+1-dimensional manifold $\mathcal{M}$ with a Lorentzian metric g_{ab} definded on it.[1] The metric locally can be transformed to the Minkowski metric, such that at any point of $\mathcal{M}$ the transformed metric equals the Minkowski metric and the first order partial derivatives with respect to the coordinates vanish. This is the mathematical implementation of the principle of equivalence. The tangent space $T_p\mathcal{M}$ at each $p \in \mathcal{M}$ is isomorphic to the Minkowski space $\mathbb{R}^{3+1}$.
In the theory of Special Relativity test particles move on straight lines. In a curved spacetime test particles are supposed to move on geodesics, the most natural generalisation of straight lines: A geodesic is a curve whose curvature in the tangent space vanishes. If such a curve is given by a parametrization $x^a = x^a(s)$ (the x^a 's are a coordinate system) it has to fulfill the
geodesic equation

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = \lambda\frac{dx^a}{ds},$$

where λ is a constant[2] and Γ^a_{bc} is the metric connection or the *Christoffel symbols*

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}).$$

One can say that the Christoffel symbols represent the gravitational field, as they imply that there is a 'force' which acts on test particles (see the geodesic equation) which leads to a deviation of the straight motion relative to the coordinates. However there is no absolute (coordinate-independent) meaning of the term 'force' in General Relativity.
The metric determines the motion of all test particles via the metric connection. The information on the curvature of the manifold is contained in the *Riemann tensor*

$$R_{abc}{}^{d} = \partial_b\Gamma^d_{ac} - \partial_a\Gamma^d_{bc} + \Gamma^e_{ac}\Gamma^d eb - \Gamma^e_{bc}\Gamma^d_{ea}.$$

This tensor does not only allow one to extract curvature data, it is also the gateway to Einstein's equations of gravitation.
From the Riemann tensor one obtains tensors of lower grade which contain reduced information on curvature. The *Ricci tensor* R_{ab} is defined as

$$R_{ab} := R_{acb}{}^{c},$$

and a further simplification occurs if one considers the *scalar curvature*

$$R := R_a{}^a.$$

[1] The signature of the metric is (−,+,+,+) throughout this text.
[2] If $\lambda = 0$ the parametrization is called *affine*.

Einstein's idea was to establish a link between the curvature of the spacetime and *all* matter or energy contained in it.

"The matter or energy determines the curved metric and the metric tells the matter how to move."

In a curved spacetime matter and energy fields are described by the *stress energy tensor* T_{ab}. Its form depends on the kind of matter field and has to be determined by correspondency considerations. In particular the stress energy tensor fulfills a divergence indentity which is the covariant generalization of both the classical conservation equations and the classical equation of motion for the field without sources:

$$\nabla^a T_{ab} \equiv 0.$$

From the curvature tensor Einstein then constructed a new tensor which fulfils an analogous identity. By setting them to be proportional he wrote down his famous equations which identify all energy and matter as causes of curvature. Since then these equations are believed to describe the nonlocal structure of space and time at scales far bigger than the Planck scale in a correct way. They are still being tested with high precision in a few experiments (see above). Till this day the physical, mathematical and philosophical interpretation of this theory presents a big challenge for everyday thinking and scientific work.
The aforementioned 'new tensor' is the *Einstein tensor* G_{ab} which is defined as

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}.$$

Its most important property is that it fulfills the *contracted Bianchi identities*

$$\nabla^a G_{ab} \equiv 0,$$

which led Einstein to the conjecture that it might be precisely this tensor that is proportional to the stress energy tensor. The proportionality factor can be determined by considering the case of a very weak field.
Einsteins' equations for a metric on $\mathcal{M}$ read

$$R_{ab} - \frac{1}{2} R g_{ab} =: G_{ab} = 8\pi T_{ab}.$$

(Throughout this text we work with relativistic units $G = 1$, $c = 1$. The factor in nonrelativistic quantities is $\frac{8\pi G}{c^2}$ instead of 8π.) During this thesis we will concentrate on stationary and axisymmetric vacuum spacetimes and on a special method to obtain exact solutions of Einstein's equations. These will simplify to a nonlinear equation for a complex scalar, the *Ernst equation*.

Chapter 1

Stationary and axisymmetric Black Holes

The aim of this chapter is to summarize mathematical concepts which are currently used to handle Black Holes and to link them with the physical background. Moreover the chain of arguments from Einstein's equations to the existing Black Hole uniqueness theorems will be made comprehensive. The argumentation roughly follows the book of Heusler [Heusler(1996)], but here the focus lies more on physical motivations rather than on mathematical strictness.
After a brief introduction of spacetime symmetries we discuss stationary and axisymmetric spacetimes in detail. This will lead us to the definition of a Killing vector field as well as to the Ernst equation and to Weyl coordinates. The latter two will turn out to be a convenient tool to handle this class of spacetimes.
As already mentioned in the introduction, there are two main pillars of the proof which is presented in this thesis. The first one is the link between the global concept of Event Horizons in a spacetime and the local concept of a Killing horizon. This link is established by the Rigidity theorems, which in their original version were found by Hawking and Carter. They are presented in section 1.3.1. Also given is a short summary of present research on this topic. The second pillar is the Linear Problem, a method to solve Einstein's equation with certain symmetries (stationarity and axisymmetry) by a boundary value problem, which is dicussed in chapters 2 and 3.

1.1 Spacetimes with symmetries and Killing vectors

Symmetries make it much easier to solve physical problems. For nonlinear equations it is often the easiest way to obtain exact nontrivial solutions. In General Relativity one need think only of the inner and outer Schwarzschild solution or the Friedmann-Robertson-Walker isotropic universes which are both highly symmetric.

We give a mathematical notion of spacetime symmetries. This will lead to the introduction of Killing vectors[1]. The arguments are taken from chapter 7.7 of [d'Inverno(1995)].

The physics behind General Relativity is invariant under arbitrary (regular) coordinate transformations which is automatically fulfilled by the use of the tensor calculus. This is the mathematical expression of the principle that non-inertial observers are also able to explore the geometry of spacetime.

1.1.1 Mathematical notion of symmetry

There may be transformations which leave the *form* of the metric tensor invariant, which means that the transformed components of g are the same functions of the new coordinates as are the untransformed components of the old ones. Such a transformation is called an *isometry.* If one considers infinitesimal coordinate transformations of the form $x'^a = x^a + \epsilon X^a$ where the vector X is the *generator* of the transformation, one can derive a mathematical condition for an isometry:
A coordinate transformation is an Isometry if and only if its infinitesimal generator X is a Killing vector, that is if the Lie derivative of g_{ab} with respect to X vanishes:

$$\mathcal{L}_X g_{ab} = \nabla_a X_b + \nabla_b X_a = 0. \tag{1.1}$$

This is the Killing equation for a Killing vector X. It allows the determination of the symmetries of a given metric. (The metric may be written down in 'inappropriate' coordinates and one may be unable to 'see' the symmetries.)
In the following chapters we will *assume* the existence of Killing fields and we will try to solve Einstein's equations under the assumption of symmetries.
The Killing fields will turn out to be the key to the characterization of Event Horizons of stationary and axisymmetric Black Holes. Therefore it is necessary to deduce some identities for vectors, Killing vectors and their corresponding forms.

1.1.2 Properties of vectors and Killing vectors

For an arbitrary differentiable vector field a, its norm $N = (a|a)$ and its twist $\boldsymbol{\omega} = \frac{1}{2} * (a \wedge \boldsymbol{d}a)$ one finds

- $\mathcal{L}_a a = 0.$
  As the Lie derivative is a partial derivative with parallel transport along a given vector field, of course, the components of a^μ remain constant on the integral curves of a. The calculation is $\mathcal{L}_a a^\mu = a^\nu \partial_\nu a^\mu - a^n u \partial_\nu a^\mu = 0$.

- $(a|\boldsymbol{d}N) = (a|\boldsymbol{d}(a|a)) = \mathcal{L}_a N = \mathcal{L}_a (a|a) \neq (!) 0.$

For Killing vectors further properties can be proved. Contrary to an arbitrary vector a, a Killing vector will be denoted by K. As by definition K fulfils the Killing

[1] Killing vectors are named after the German mathematician Wilhelm Killing (1847-1923) who is said to have developed the theory of Lie groups independently from Sophus Lie. (http://en.wikipedia.org/)

equation, K is always differentiable. The norm of K will again be called N. Note that for manifolds with vanishing torsion, the Christoffel symbols are symmetric in their lower indices. This implies that
both the Lie derivative and the exterior derivative can be calulated with any derivative operator ([Wald(1984)], pp 429, 441).

- $\mathcal{L}_K g_{ab} = \partial_a K_b + \partial_b K_a = 0$, (the Killing equation).
- $\boldsymbol{\mathcal{L}}_K = \boldsymbol{d} \circ \boldsymbol{i}_K + \boldsymbol{i}_K \circ \boldsymbol{d}$ (Proof omitted).
- $\mathcal{L}_K |g| = 0 : \quad \mathcal{L}_K |g| = K^a \partial_a |g| = K^a \triangle_{\mu\nu} \partial_a g^{\mu\nu} = K^a \triangle_{\mu\nu} \nabla_a g^{\mu\nu} = 0.$
- $\mathcal{L}_K N = 0 : \quad \mathcal{L}_K (K|K) = K^\mu \partial_\mu K^\nu K_\nu = K^\mu (K_\nu \partial_\mu K^\nu + K^\nu \partial_\mu K_\nu) = 2K^\mu K^\nu \partial_\mu K_\nu = -2K^\mu K^\nu \partial_\mu K_\nu = 0.$[(2)]
- The Hodge dual commutes with the Lie derivative: $\boldsymbol{\mathcal{L}}_K * \boldsymbol{\alpha} = * \boldsymbol{\mathcal{L}}_K \boldsymbol{\alpha}$. This is a straight forward calculation.
- $\nabla_{(\mu)} K_{(\mu)} = 0$: part of the Killing equation.
- $\nabla_\mu K^\mu = 0 : \quad \sum_a \nabla_a K_a = 0 = \sum_a \nabla_a g_{a\mu} K^\mu = \sum_a g_{a\mu} \nabla_a K^\mu = \nabla_\mu K^\mu.$
- $\boldsymbol{d}^\dagger K = 0 : \quad \boldsymbol{d}^\dagger K = -\frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} K^\mu) = -\frac{1}{\sqrt{|g|}} \nabla_\mu (\sqrt{|g|} K^\mu) = -\nabla_\mu K^\mu = 0.$
- The Lie derivative of the twist of K vanishes, $\boldsymbol{\mathcal{L}}_K \boldsymbol{\omega}_K = 0 :$
  $\boldsymbol{\mathcal{L}}_K \boldsymbol{\omega}_K = -\frac{1}{2} * \Big(K \wedge \boldsymbol{d}^\dagger (K \wedge \boldsymbol{d}K) + \boldsymbol{d}^\dagger (K \wedge K \wedge \boldsymbol{d}K) \Big)$
  $= -\frac{1}{2} * \Big(K \wedge (\boldsymbol{d}^\dagger K \wedge \boldsymbol{d}K - K \wedge \boldsymbol{d}^\dagger \boldsymbol{d}K) \Big) = \frac{1}{2} K \wedge K \wedge \boldsymbol{d}^\dagger \boldsymbol{d}K = 0.$

1.1.3 Stationary and axisymmetric (s&a) spacetimes

Stationary final states of Black Holes are 'invariant under time translations'. However it turns out that there are spacetimes which allow stationary observers only in some regions.[3] When speaking of stationarity we thus have 'asymptotic stationarity' in mind. The aim of this work is to find spacetimes which contain a stationary Black Hole. One may therefore assert a symmetry in analogy to the Newtonian theory. However the notion of symmetries in General relativity cannot be one-to-one, as one considers a spacetime rather than space and time. There is no absolute time, and hence the concept of time translation is obsolete in the strict sense. As we know that symmetries are related to Killing fields we define a spacetime to be stationary if it is invariant under a transformation with asymptotically timelike orbits, that is if there exists an asymptotically timelike Killing vector ξ. Let us now have a short look at asymptotic flatness and stationarity.

[2]For vector fields a this is in general not true. Consider for example the manifold $\mathbb{R}^2$ with cartesian coordinates (x, y) and the vector $a^\mu = ((x, 0)^T)^\mu$.

[3]Consider for example the Kerr Black Hole. In the ergoregion the asymptotic timelike Killing vector is spacelike and hence no observer at infinity can ever see a stationary test particle in this region.

Asymptotically flat s&a spacetimes There is an intuitive picture of asymptotic flatness: Far away from matter distributions the metric tensor becomes flat. But given a special coordinate system, how can one know where one has to look for 'infinity'? And how fast should the metric converge to the Minkowski metric? There are two ways to handle asymptotical flatness:

- There is a precise definition via Penrose diagramms ([Wald(1984)], chapter 11).
- If a coordinate system with well known properties is used it will be clear how 'infinity' can be reached.

In both approaches one follows future directed causal geodesics.
Fortunately the Weyl coordinates are a regular coordinate system on the whole domain of outer communication[4] of $\mathcal{M}$ and spatial infinity is well defined.
Now let us return to stationary spacetimes:
Definition: An asymptotically flat spacetime is called stationary if it contains an asymptotically timelike Killing vector field ξ.
This allows an observer to remain stationary, if he moves in a way that his time direction is always parallel to the trajectories of ξ. The definition admits regions where ξ is spacelike or null. In these regions an observer cannot remain stationary.

On a first glance there is no reason to assume the existence of further symmetries for stationary Black Holes. However the Rigidity theorem (section1.3.1) states that for a certain class of spacetimes - stationary Black Holes - there must exist an additional axial symmetry. For this reason axisymmetry has to be taken into account from the beginning on:

Definition: A spacetime is called axisymmetric if it is invariant und the action of a 1-parameter group $\mathcal{G} = SO(2)$ and if there is an nonempty set of fixed points of $\mathcal{G}$, the rotation axis.
This definition introduces a further Killing vector η which is spacelike everywhere exept along the axis, where it is zero.

Definition: A spacetime is called 'stationary and axisymmetric' (abbreviated: s&a) if the Killing vectors which generate the above symmetries commute with each other.

1.1.4 The cirularity theorem

In order to find coordinates which reflect the symmetry it is necessary to gain information about possible foliations of spacetime. In the case of s&a spacetimes we are interested in knowing whether the two Killing fields are integrable, i.e. whether there is a hypersurface which is at each point spanned by the Killing vectors. In our case this is ensured by the circularity theorem. It is also known as the theorem of Kundt and Trümper.

[4] The domain of outer communication is the part of a spacetime from where causal geodesics can escape to infinity.

Theorem: Stationary and axisymmetric spacetimes admit 2-spaces orthogonal to the Killing fields iff

$$\xi \wedge \eta \wedge R(\eta) = 0 = \xi \wedge \eta \wedge R(\xi), \tag{1.2}$$

where $R(X)$ is the Ricci-tensor contracted with a vector X or the corresponding form:

$$R(X)_\mu = R_{\mu\nu} X^\nu. \tag{1.3}$$

As we intend to consider a vacuum solution, $R_{\mu\nu} = 0$, and the foliation exists.

As already mentioned in the introduction, spacetimes with this high degree of symmetry are of interest for both theoretical physicists and astronomers, as they are considered as possible final states of astrophysical bodies and they they can be handled with moderate effort. There is a number of s&a solutions available and there are also uniqueness theorems for vacuum spacetimes, the *Black Hole uniqueness theorems*.
All results of this thesis are about spacetimes that are stationary and axisymmetric and that are asymptotically flat.

1.2 Weyl coordinates and the Ernst equation

For s&a spacetimes there will be a remarkably simple - but still hard to solve - nonlinear equation for a complex scalar f which can be derived from Einstein's equations. There are coordinates for any s&a vacuum spacetime which are valid on the whole domain of outer communication. In one of these maps, the Weyl coordinates, the metric tensor takes a simple form. We start our investigation with a corollary from Heuslers book ([Heusler(1996)], p. 38):
Corollary: Consider an asymptotically flat, stationary and axisymmetric spacetime $(M, {}^{(4)}g)$ admitting a foliation by integrable 2-surfaces orthogonal to the Killing fields. Then locally $M = \perp \times \bigcirc$ and ${}^{(4)}g = \tau \otimes \sigma$ (See also page 70). We call $\perp$ the orthogonal and $\bigcirc$ the orbit manifold. Each of them has locally a separate metric. For σ we have coordinates t, ϕ which are adapted to the Killing vector fields:

$$-\xi_i \xi^i = -g_{tt} =: e^{2U}; -\eta_i \xi^i = -g_{\phi t} =: a e^{2U}; \eta_i \eta^i = g_{\phi\phi} =: -a^2 e^{2U} + W^2 e^{-2U}. \tag{1.4}$$

We have written down σ in terms of the three functions e^{2U}, a and W. Note that W is the determinant of σ.
Now we fokus our attention to $\perp$. One can show (for example [Ansorg(1998)]) that it is always possible to find coordinates in which τ is conformally equivalent to the flat metric, $\tau = \Omega^2 \mathbb{I}$. Now consider a change in coordinates which does not change the form of τ. Denote the old coordinates by (x,y) and the new ones by (a,b), and the conformal factors Ω^2 and ω^2, respectively. Then we can write down the transformation law for the metric tensor:

$$\frac{1}{\omega^2}\delta^{\alpha\beta} = \tau'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x\gamma}\frac{\partial x'^\beta}{\partial x\delta}\tau^{\gamma\delta} = \frac{\partial x'^\alpha}{\partial x\gamma}\frac{\partial x'^\beta}{\partial x\delta}\frac{1}{\Omega^2}\delta^{\gamma\delta}.$$

If J denotes the Jacobi matrix of the transformation, the upper equation states that $J \cdot J^t = \mathbb{I}$ which reads

$$J = \begin{pmatrix} a_{,x} & a_{,y} \\ b_{,x} & b_{,y} \end{pmatrix}, \quad J \cdot J^t = \begin{pmatrix} a_{,x}^2 + a_{,y}^2 & a_{,x} b_{,x} + a_{,y} b_{,y} \\ a_{,x} b_{,x} + a_{,y} b_{,y} & b_{,x}^2 + b_{,y}^2 \end{pmatrix} \propto \mathbb{I}. \tag{1.5}$$

Note that any two C^1-functions satisfying

$$a_{,x} = b_{,y}\,; \qquad a_{,y} = -b_{,x} \tag{1.6}$$

leave the form of the metric unchanged.

Now consider the function $\rho = W(x,y) := \sqrt{-|\sigma|}$. The Einstein equation for ρ turns out to be $\triangle^{\perp}\rho = 0$, and one can argue that for this reason ρ has no critical points in the domain of outer communications and hence can be chosen as a coordinate on Σ (see [Heusler(1996)] p. 39). Now consider the 'conjugate harmonic function' ζ which satisfies (1.6) (with $a = \rho$ and $b = \zeta$) and thus $\triangle^{\perp}\zeta = 0$. We choose ρ and ζ as coordinates on Σ and have then $(\boldsymbol{d}\rho|\boldsymbol{d}\rho)^{\perp} = (\boldsymbol{d}\zeta|\boldsymbol{d}\zeta)^{\perp}$ and $(\boldsymbol{d}\rho|\boldsymbol{d}\zeta)^{\perp} = 0$. The metric has now the following form:

$$g = \begin{pmatrix} e^{-2U}e^{2k} & 0 & 0 & 0 \\ 0 & e^{-2U}e^{2k} & 0 & 0 \\ 0 & 0 & e^{-2U}\rho^2 - e^{2U}a^2 & -e^{2U}a \\ 0 & 0 & -e^{2U}a & -e^{2U} \end{pmatrix}. \tag{1.7}$$

This is the metric tensor of a stationary, axisymmetric spacetime in Weyl's canonical coordinates. Troughout the text we will name them *Weyl coordinates.*

(The conformal factor ω^2 of $g^{\perp}$ was split into e^{2U} and the function e^{2k}.)

1.2.1 The derivation of the Ernst equation

The line element of a stationary and axisymmetric spacetime in Weyl coordinates has the form

$$ds^2 = e^{-2U}(e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\phi^2) - e^{2U}(dt + a d\phi)^2. \tag{1.8}$$

It is a long, but straightforward calculation to write down Einstein's equations with respect to this special line element. After the introduction of complex coordinates $z = \rho + i\zeta$ the equations read for $W \equiv \rho$ [Meinel(1991)]:

$$\begin{aligned} (\rho U_{,z})_{,\bar{z}} + (\rho U_{,\bar{z}})_{,z} &= -\frac{e^{4U}}{\rho} a_{,z}\, a_{,\bar{z}} \\ \Big(\frac{e^{4U}}{\rho} a_{,z}\Big)_{,\bar{z}} + \Big(\frac{e^{4U}}{\rho} a_{,\bar{z}}\Big)_{,z} &= 0 \\ k_{,z} &= 2\rho (U_{,z})^2 - \frac{e^{4U}}{\rho}(a_{,z})^2. \end{aligned} \tag{1.9}$$

Once one has determined U and a from the first two equations, k can be calculated from the third by a line integration because the first to equations contain the integrability conditions for the k-equation. The second equation can formally be solved by introducing a new variable b and by setting

$$b,_z := -i\frac{e^{4U}}{\rho}a,_z .$$

The second equation then reads

$$(\rho e^{-4U}b,_z),_{\bar{z}} + (\rho e^{-4U}b,_{\bar{z}}),_z = 0.$$

Now one defines the complex quantity

$$f := e^{2U} + ib$$

and finds that the remaining two equations are equivalent to the *Ernst equation*

$$\Re f\Big((\rho f,_{\bar{z}}),_z + (\rho f,_z),_{\bar{z}}\Big) = 2\rho f,_z f,_{\bar{z}} \tag{1.10}$$

or - in ρ and ζ:

$$\Re f\Big(f,_{\rho\rho} + f,_{\zeta\zeta} + \frac{1}{\rho}f,_\rho\Big) = (f,_\rho^2 + f,_\zeta^2). \tag{1.11}$$

f is usually called *Ernst potential.* All three metric functions can be determined from f:

- e^{2U} is the real part of f.
- a can be determined by a line integral: $a,_\rho = \rho e^{-4U}b,_\zeta$; $\qquad a,_\zeta = -\rho e^{-4U}b,_\rho$.
- k also is obtained by a line integral:

$$k,_\rho = \tfrac{\rho}{4}e^{-4U}(e^{2U},_\rho^2 + e^{2U},_\zeta^2 + b,_\rho^2 + b,_\zeta^2); \quad k,_\zeta = \tfrac{\rho}{2}e^{-4U}(e^{2U},_\rho e^{2U},_\zeta + b,_\rho b,_\zeta).$$

The latter two formulas are just the ones written down above, but in ρ and ζ-coordinates.

1.3 Black Holes

Event Horizons in General Relativity can be defined globally and locally as well. The Black Hole region is the interior of the Event Horizon.

Event Horizons as trapped surfaces A brief summary will explain how the concept of an Event Horizon as 'region of no escape' can be formulated more precisely. An asymptotically flat spacetime $\mathcal{M}$ without *naked singularities* posesses a *Black Hole region* if the *causal past* J^- of *future null infinity* $\mathcal{J}^+$ does not cover the hole spacetime.[5] This means that there is a region in $\mathcal{M}$ from where no causal (= timelike or null) curve can reach the 'outside', the domain of outer communication. This region

$$\mathcal{B} := \mathcal{M} \setminus J^-(\mathcal{J}^+) \tag{1.12}$$

[5] These terms are defined in [Wald(1984)], chapter 8.

is called the *Black Hole region* and its boundary $\mathcal{H}$ in $\mathcal{M}$

$$\mathcal{H} := \partial\Big(\mathcal{M} \setminus J^-(\mathcal{J}^+)\Big) = \partial(J^-(\mathcal{J}^+)) \cap \mathcal{M} \tag{1.13}$$

is called the *Event Horizon* or just *horizon*. This abstract definition only makes use of causality considerations.

Killing Horizons There is also a local concept to define an Event Horizon which has at a first glance not much to do with the abstract global formulation above. For stationary spacetimes it is possible to define the horizon locally: A Killing horizon is a null hypersurface with a Killing field normal to it. To be more precise:
Definition: Consider the set of points where a given Killing vector χ is null but not zero. Any union $\mathcal{H}(\chi)$ of connected components of the hypersurfaces $(\chi|\chi) = 0,\ \chi \neq 0$ is called a Killing Horizon.

The Rigidity Theorem (cf. 1.3.1) states that for analytic spacetimes the Event Horizon *is* a Killing horizon. Of course it is much easier to obtain the behaviour of metric functions with a locally defined horizon. The theorem even states that a further axial symmetry exists und thus it allows one to classify *all* stationary asymptotically flat vacuum Black Hole spacetimes.

In particular it distingiushes between rotating and non-rotating Black Holes.
A *non-rotating Black Hole* is described by the fact that the horizon-defining Killing vector coincides with stationary Killing field ξ, which refers to time translation symmetry in the outside. If there exist two other axial Killing vectors, the spacetime is spherically symmetric an thus is static in the outside. This is the statement of Birkhoff's theorem.
If the spacetime is asymptotically flat and there is a Killing Horizon, it is called a 'rotating Black Hole spacetime' or '*rotating Black Hole*'. Another consequence of asymptotic flatnes is the fact that ξ and η commute, $[\xi, \eta] = 0$, which was shown by Carter [Carter(1970)].

The Killing horizon of a rotating Black Hole The Killing field which defines the horizon must be a linear combination of ξ and η. From the Rigidity Theorem (1.3.1) we know that in the non-static case we do have two Killing vectors and the Generator of the horizon is given by

$$\chi := \xi + \Omega_H \eta. \tag{1.14}$$

This the generator of the horizon of a rotating Black Hole which will be used from now on. Ω_H is called the 'angular velocity' of the horizon. The horizon $\mathcal{H}(\chi)$ itself is invariant under symmetry transformations generated by ξ and η. They both are tangent to $\mathcal{H}(\chi)$. Since a tangential null vector is also normal to $\mathcal{H}$[6], we can calculate the angular velocity Ω_H on the horizon.

$$(\chi|\chi)|_{\mathcal{H}} = 0, \qquad (\chi|\xi)|_{\mathcal{H}} = 0, \qquad (\chi|\eta)|_{\mathcal{H}} = 0. \tag{1.15}$$

[6]It is tangential in the pictorial sense and normal with respect to the inner product.

This provides us two formula for Ω_H:

$$\Omega = -\frac{(\xi|\xi)}{(\xi|\eta)} = -\frac{(\xi|\eta)}{(\eta|\eta)}, \qquad \Omega_H = \Omega|_{\mathcal{H}}. \tag{1.16}$$

$$\text{Thus,} \qquad (\xi|\xi)(\eta|\eta) - (\xi|\eta)^2 \overset{\mathcal{H}}{=} 0.$$

From now on, Ω_H denotes the value of the function Ω on a given point of $\mathcal{H}$. It is constant on $\mathcal{H}$, as will be shown later on. The Killing-2-form of a rotating Black Hole is defined as the wedge product of ξ and η, and for its norm σ one finds

$$\begin{aligned}
\sigma &:= 2(\xi \wedge \eta|\xi \wedge \eta) \qquad (1.17)\\
&= 2(\frac{1}{2}(\xi \otimes \eta - \eta \otimes \xi)|(\frac{1}{2}(\xi \otimes \eta - \eta \otimes \xi))\\
&= \frac{1}{2}(2 \cdot (\xi|\xi)(\eta|\eta) - 2 \cdot (\xi|\eta)^2)\\
&\overset{\mathcal{H}}{=} 0.
\end{aligned}$$

Note that $\sigma = -\rho^2$ (cf. 4.7), which means that on $\mathcal{H}$ one has always

$$\rho \overset{\mathcal{H}}{=} 0. \tag{1.18}$$

Properties of Ω on the horizon Among several properties of Ω on $\mathcal{H}$, we will give a proof of the outstanding feature $\boldsymbol{d}\Omega \overset{\mathcal{H}}{=} 0$.
$\Omega_H\eta$ is a Killing vector: Consider the Lie-derivative of the metric tensor with respect to the Killing vector χ,

$$0 = \mathcal{L}_\chi g_{ab} = \mathcal{L}_\xi g_{ab} + \mathcal{L}_{\Omega_H\eta} g_{ab} = \mathcal{L}_{\Omega_H\eta} g_{ab}.$$

The Lie-derivative of the metric with respect to $\Omega_H\eta$ vanishes, which means that $\Omega_H\eta$ is a Killing vector.
$\mathcal{L}_\eta\Omega_H$ vanishes: First of all let us write down the Killing equation for $\Omega_H\eta$:

$$\begin{aligned}
0 &= \mathcal{L}_{\Omega_H\eta} g_{ab} = \nabla_b \Omega_H \eta_a + \nabla_a \Omega_H \eta_b\\
&\overset{Leibnitzrule}{=} \Omega_H \cdot (\nabla_b\eta_a + \nabla_a\eta_b) + \eta_a \nabla_b \Omega_H + \eta_b \nabla_a \Omega_H\\
&= 0 + \eta_a \partial_b \Omega_H + \eta_b \partial_a \Omega_H.
\end{aligned}$$

The Killing property of η and the equality of partial and covariant derivative for scalars have been used in the last step. Now we contract the last line with $\eta^a\eta^b$:

$$0 = \eta^a\eta^b(\eta_a \partial_b \Omega_H + \eta_b \partial_a \Omega_H)$$

$$0 = (\eta|\eta)(\eta^b\partial_b + \eta^a\partial_a)\Omega_H$$

$$0 = \eta^a \partial_a \Omega_H = \mathcal{L}_\eta \Omega_H. \tag{1.19}$$

The gradient of Ω*,* $\boldsymbol{d}\Omega$*, vanishes on* $\mathcal{H}$*:* We contract the Killing equation for $\Omega_H\eta$ with η^a:

$$0 = (\eta|\eta)\partial_b\Omega_H + \eta_b\eta^a\partial_a\Omega_H = (\eta|\eta)\partial_b\Omega_H + \eta_b\mathcal{L}_\eta\Omega_H$$

As $(\eta|\eta)$ does not vanish off the axis, we obtain

$$\partial_b\Omega_H = 0$$

or, in an invariant way,

$$\boldsymbol{d}\Omega \overset{\mathcal{H}}{=} 0. \tag{1.20}$$

The quantity $\Omega = -\frac{(\xi|\xi)}{(\xi|\eta)} = -\frac{(\xi|\eta)}{(\eta|\eta)}$ is constant on the 3-dimensional hypersurface $\mathcal{H}$. The rigid rotation of the horizon has lead to the name 'Rigidity theorem', as this statement is a direct consequence of the theorem.

1.3.1 The Rigidity Theorems and their consequences

The theorems and sketches of their proofs can be found in Heusler's book, chapter 6.2. [Heusler(1996)]. We will briefly recapitulate them here and provide a few comments about the state of the full proof.

Strong Rigidity theorem
(1) The event horizon of a stationary vacuum Black Hole spacetime is a Killing horizon, provided that spacetime is analytic, the fundamental matter fields obey well behaved hyperbolic equations and the stress-energy tensor fulfil the weak energy condition.

(2) The Horizon Killing field χ either coincides with the stationary Killing field ξ, or the spacetime admits at least one axial Killing field

A more precise version with a corrected proof can be found in [Chruściel(1996)], with analyticity still required. This fundamental result relates the global concept of an Event Horizon with that of a Killing horizon in a stationary spacetime. The main obstacle for a wider application of this theorem is the required analyticity of the metric, especially in the case of ergoregions this is difficult to show.
There is a weaker version of the above theorem, where the existence of a further Killing vector is assumed, the weak Rigidity Theorem:
Weak Rigidity theorem
Consider a circular spacetime, $\lambda = \xi + \omega\eta$, $\omega = -\frac{(\xi|\xi)}{(\xi|\eta)}$ and $\mathcal{S}[\lambda]$ the surface $(\lambda|\lambda) = 0$. Then ω is constant on $\mathcal{S}[\lambda]$, λ is a Killing vector on $\mathcal{S}[\lambda]$ and $\mathcal{S}[\lambda]$ is a stationary null surface: $\mathcal{S}[\lambda] = \mathcal{H}[\lambda]$.

Vacuum s&a spacetimes are always circular (See 1.1.4). As we only calculate solutions of the Ernst equation with the help of the Linear Problem we can apply the weak Rigidity Theorem. This is a big advantage, as the analyticity of the metric is no longer required. It states that the Killing Horizon which is generated by $\lambda = \xi + \omega\eta$ is an Event Horizon. However it gives us no classification of stationary spacetimes. The existence of an axial symmetry has to be assumed.
Recently, I. Rácz published a version of the theorem which requires the existence

of a smooth rather than an analytic spacetime [Rácz(2007)]. The degenerate case is still excluded.

1.3.2 The topology of the horizon

If we intend to solve the Ernst equation via a boundary value problem we have to find reasonable boundary conditions first. Here the question of the connectedness of the horizon is tackled.

The topology theorem Black Holes of a certain symmetry may still differ in their topological properties. One would like to know whether there are fixed topological attributes of the horizon like its Euler characteristic or the degree of its connectedness.
For stationary spacetimes the *topology theorem*, which goes back to Hawking, is available. Its proof is considered complete [Heusler(1998)], ch. 1.1; [Chruściel(1996)]. The aim is to show that any connected component of an Event Horizon has the topology $\mathbb{R} \times S^2$ and that the domain of outer communication is simply connected. There is a generalized result available, which was proved by Galloway [Galloway(1995)].

Topology of the domain of outer communication, (G. Galloway) Consider an asymptotically flat globally hyperbolic spacetime with its null infinity satisfying the regularity condition. If

$$R_{\mu\nu}X^\mu X^\nu \geq 0$$

for all null vectors X^μ, then every globally hyperbolic domain of outer communication is simply connected.
The topology of the horizon is a consequence of this theorem, but is has been proven earlier by P.T. Chruściel and R.M. Wald [Chruściel and Wald(1994)]:
The asymptotically timelike Killing vector generates a one-parameter group of isometries. Thus there is a family of achronal, asymptotically flat slices of the domain of outer communication. Each slice intersects the Event Horizon, and the intersection is homeomorphic to a 2-sphere. This holds for any value of the parameter of the isometry.
Now that we know about the topological properties of the horizon we can apply these results to Weyl coordinates.

The Event Horizon in Weyl coordinates The horizon it is invariant under the action of the two Killing vectors. There is only one direction in the tangent space of a $p \in \mathcal{H}$ which also lies in the tangent space of $\perp$ in p. Locally, the horizon will be an interval in Weyl coordinates. We will see in chapter 4 that $\rho = 0$ on $\mathcal{H}$. There we will prove that the gradient of ζ on a connected component of the horizon will be zero if and only if the horizon is degenerate. In the non-degenerate case we can conclude that a non-degenerate single horizon is an interval on the ζ-axis. This is a condition for the boundary value problem of the Ernst equation.

1.4 Black Hole uniqueness

We briefly summarize the available uniqueness theorems for Black Holes. The gap in the Rigidity Theorem is taken into consideration. Then we list the arguments that are necessary to obtain a uniqueness statement for the extreme Kerr Black Hole.
The most remarkable consequence of the uniqueness theorems is that they make very precise statements about the final states of isolated Black Holes.

The established uniqueness theorems The Rigidity theorem classifies the stationary vacuum Black Holes: *Non-degenerate stationary analytic electrovac Black Holes are either static, or axisymmetric.* (This is the rough version of the theorem from [Chruściel(1996)].) This allows one to consider only spacetimes with Killing horizons. From this point the following uniqueness theorems for non-degenerate electrovac spacetimes have been proved (same reference as above):

The Reissner-Nordström non-degenerate Black Holes exhaust the family of static non-degenerate electrovac Black Holes.

The non-degenerate Kerr-Newman Black Holes exhaust the family of non-degenerate stationary and axisymmetric electrovac Black Holes.

For the uncharged Kerr Black Hole there is a constructive uniqueness proof by Meinel and Neugebauer which is explained in detail in chapter 3.

Outline on the results of this thesis In chapter 4 the following extension to the already existing constructive uniqueness proofs will be given:

- The extreme Kerr Black Hole is the only degenerate stationary and axisymmetric vacuum Black Hole with connected horizon.

To make a uniqueness statement for a stationary family of Black Holes out of this fact one first has to find a version of the Rigidity theorem that also holds in the case of degenerate horizons. Then the statement would be

- The Kerr Black Holes exhaust the family of stationary and axisymmetric vacuum Black Holes with single horizon.

First we will show that for s&a vacuum spacetimes a simply connected degenerate Horizon is a point $(0, \zeta_0)$ on the ζ-axis in Weyl coordinates. Using the Linear Problem we then construct a unique solution of the Ernst equation for such a horizon. With an extended version of the Rigidity theorem one could conclude that stationary vacuum Black Holes even with degenerate horizon are either static or axisymmetric. In case of axisymmetry and a simply connected Event Horizon the statement of this thesis would follow.

Chapter 2

Inverse Scattering Methods for nonlinear equations

This short chapter shall briefly summarize the basic ideas of the Inverse Scattering Method which can be used to solve nonlinear equations. Further on, the link between this method and the work of G. Neugebauer and R. Meinel shall be explained. We give an outline of how Bäcklund transformations and Riemann-Hilbert-problems are used to obtain stationary and axisymmetric solutions of Einstein's equations.

We intend to embed the chain of arguments which lead to the constructive uniqueness proofs which were found by Neugebauer and Meinel [Neugebauer and Meinel(2003)] in a wider context. Many details on this topic can be found in the above cited article and in the book [Meinel(1991)]. Here only the rough structure shall be sketched in order to link several topics that do not seem to have much in common at first glance.

2.1 Introductory example

The origin of the Inverse Scattering Method (ISM) The term "Inverse Scattering Method" itself has nothing to do with nonlinear equations. It arises in quantum mechanics if one wants to solve the inverse problem of the Schrödinger equation, that is to determine the classical potential $V(x)$ from the scattering data. These data include information which are determined by scattering experiments. They consist of the energy-dependent reflection coefficient, the energy eigenvalues and the asymptotic behaviour of the energy eigenfunctions. If V varies with time, $V = u(x,t)$, it is in general impossible to find analytic solutions for the time dependent scattering data. However there are remarkable exeptions, one of them is described in [Novikov et al.(1984)], Page 23: If the time dependence of the potential is given by the Korteweg-de-Vries (KdV) equation,[1]

$$V_t \equiv u_t = 6u \cdot u_x - u_{xxx}, \tag{2.1}$$

[1] The KdV equation has its origin in hydrodynamics where it is a model for solitons in flat water basins.

$$
\begin{tikzcd}
\boldsymbol{S(0)} \arrow[r, "\text{time evolution}"] & \boldsymbol{S(t)} \arrow[d, "\text{ISM}"', "S(t) \Longrightarrow u(x,t)"] \\
\boldsymbol{u(x,0)} \arrow[u, "V = u(x,0)", "\text{S-equ.}"'] \arrow[r, dashed, "\text{KdV}", "\text{Cauchy-problem}"'] & \boldsymbol{u(x,t)}
\end{tikzcd}
$$

Figure 2.1: Solving the KdV equation using linear problems.

a rather simple solution for the evolution of the scattering data can be obtained. This allows one to solve the Cauchy problem (evolution from $u(x,0)$ to $u(x,t)$) for the KdV equation: First one solves the direct problem for the one dimensional Schrödinger equation for the potential $V(x) = u(x,0)$. This provides the scattering data $S(0)$. Then one calculates the data $S(t)$ at a later time by solving the linear time-dependent Schrödinger equation. The third step is the inverse problem: Calculating $u(x,t) \equiv V(x,t)$ from the scattering data $S(t)$ gives a new solution of the KdV equation. Thus one can find the nonlinear time evolution of a soliton by solving in turn three linear differential equations.

2.2 The ISM and Linear Problems

Starting from this important example one wishes to know wether there is a more systematic way to arrive at linear equations that can help to solve the Cauchy problem for associated nonlinear equations. Some details can be found in [Novikov et al.(1984)], chapter III.
For a wide class of nonlinear partial differential equations there is a linear matrix problem whose integrability condition (the equality of second order partial derivatives) can be made equivalent to the nonlinear equation. This is achieved by the introduction of a further complex variable λ, the so-called spectral parameter. After one has determined the structure of the Linear Problem one can write down the integrability condition. This is a power series in λ, and the coefficients which have to be zero for arbitrary but fixed x and t will be directly related to a differential equation. The focus lies on those problems in which the related differential equations in (x,t) are *nonlinear*. Sometimes complex variables $z = x + \mathrm{i}t$, $\bar{z} = x - \mathrm{i}t$ are used instead of x and t. A typical structure of a Linear Problem as it is widely used is the following:

$$\boldsymbol{\Phi}_{,z} = \boldsymbol{U}\boldsymbol{\Phi}; \qquad \boldsymbol{\Phi}_{,\bar{z}} = \boldsymbol{V}\boldsymbol{\Phi}, \tag{2.2}$$

where $\boldsymbol{\Phi}$, $\boldsymbol{U}$ and $\boldsymbol{V}$ are complex 2×2-$\mathbf{C}^2$-matrices depending on x, t and λ. The corresponding integrability condition is

$$\Big(\boldsymbol{U}\boldsymbol{\Phi}\Big)_{,\bar{z}} = \Big(\boldsymbol{V}\boldsymbol{\Phi}\Big)_{,z} \quad \Longleftrightarrow \quad \boldsymbol{U}_z - \boldsymbol{V}_{\bar{z}} + [\boldsymbol{U}, \boldsymbol{V}] = 0. \tag{2.3}$$

The procedure of the ISM is to consider $\boldsymbol{\Phi}$ for arbitrary but fixed $z, \bar{z}$ as holomorphic functions in λ or K and to determine the coefficients of the Linear Problem afterwards from $\boldsymbol{\Phi}(z, \bar{z}, \lambda)$.

2.3 How to solve a Linear Problem - an outline

We briefly want to explain how several techniques are used to solve a Linear Problem. We focus on three ways which are used by Meinel and Neugebauer [Neugebauer and Meinel(2003)]:

- Integration of the Linear Problem along certain paths
- The polynomial Bäcklund transformation
- The Riemann-Hilbert problem

2.3.1 Integration

By line integration the Linear Problem can be solved on special curves in the $(z, \bar{z})$-plane as there may be boundary conditions which simplify the calculation. One gets $\mathbf{\Phi}(z, \bar{z}, \lambda)$ and the coefficients of the Linear Problem on a certain curve. This procedure is called the *direct problem* of the ISM ([Neugebauer and Meinel(2003)]). One then may be able to extend the solution to all values of z and $\bar{z}$ by using other techniques and by comparison with the solution on the curve. In chapter 3 we will describe a detailed calculation of this kind, which will lead to the Kerr metric. As this calculation gives the solution on a certain curve only, other techniques have to be used in order to determine the matrix $\mathbf{\Phi}$ everywhere. Thus this method is useful only under special circumstances.

2.3.2 The polynomial method of the Bäcklund Transformation

The term "Bäcklund Transformation" is used in this text to desribe a transformation of a function which fulfills a certain differential equation, to another function in order to obtain a solution to a perhaps different equation. As an example[2] we consider the ordinary Laplace equation: Let u be a solution of the Laplace equation on $\mathbb{R}^2$. Then a function v related to u by the Cauchy Riemann equations

$$u_{,x} = v_{,y}\,; \qquad\qquad u_{,y} = -v_{,x}$$

is again a solution of the Laplace equation, due to the integrability condition for u.

No systematic way to find such transformations is known yet. However it is possible to find Bäcklund transformations for some classes of Linear Problems. This can be done by the polynomial method [Meinel(1991)], chapter 4.3.:
Given a known solution $\mathbf{\Phi}_0$ of the Linear Problem, which may be obtained by considering a trivial solution of the nonlinear equation,[3] define a new matrix $\mathbf{\Phi}$ by

$$\mathbf{\Phi} = \boldsymbol{T}\mathbf{\Phi}_0,$$

[2] cited from `http://en.wikipedia.org/wiki/B%C3%A4cklund_transformation` (*sic*!)
[3] For example the Minkowski space as trivial solution of the Ernst equation.

$$\begin{tikzcd}
\boldsymbol{\Phi}_0 \arrow[r, "\text{Polynom Bäckl.}"] & \boldsymbol{\Phi} \arrow[d, "\text{inverse L.P.}"] \\
\boldsymbol{f}_0 \arrow[u, "\text{L.P.}"] \arrow[r, dashed, "\text{unknown}", "\text{Bäcklund transform}"'] & \boldsymbol{f}
\end{tikzcd}$$

Figure 2.2: Scheme of the polynomial Bäcklund Transformation, f fullfills the nonlinear equation.

where

$$\boldsymbol{T} = \sum_{j=0}^{N} \frac{\boldsymbol{C}_j(\rho,\zeta)}{\lambda^j} = \boldsymbol{C}_0 + \boldsymbol{C}_1 \cdot \frac{1}{\lambda} + ... + \boldsymbol{C}_N \cdot \frac{1}{\lambda^N}.$$

From this ansatz one can calculate new matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ wich lead to new solutions of the same nonlinear equation, which is the main consequence of Theorem 4.1 in [Meinel(1991)]:
Theorem 4.1 from "Solitonen": Let $\boldsymbol{U}_0$ and $\boldsymbol{V}_0$ be rational matrix valued functions of the complex spectral parameter λ , then $\boldsymbol{U}$ and $\boldsymbol{V}$ are again rational matrix valued functions, and $\boldsymbol{U}$ has exactly the same poles as $\boldsymbol{U}_0$. The same holds for $\boldsymbol{V}$ and $\boldsymbol{V}_0$.

The matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ are now

$$\begin{aligned}
\boldsymbol{U} &\equiv \boldsymbol{\Phi}_x \boldsymbol{\Phi}^{-1} &&= \boldsymbol{T}_x \boldsymbol{T}^{-1} + \boldsymbol{T}\boldsymbol{U}_0\boldsymbol{T}^{-1} \\
\boldsymbol{V} &\equiv \boldsymbol{\Phi}_t \boldsymbol{\Phi}^{-1} &&= \boldsymbol{T}_t \boldsymbol{T}^{-1} + \boldsymbol{T}\boldsymbol{V}_0\boldsymbol{T}^{-1}.
\end{aligned}$$

Once one has determined $\boldsymbol{U}$ and $\boldsymbol{V}$ one can calculate the Bäcklund transformed solution from the coefficients of the transformed Linear Problem.
An example for the polynomial method is the derivation of the Kerr metric by a Bäcklund transform of the Minkowski space [Neugebauer and Meinel(2003)]. The Linear Problem is integrated along the axis and the resulting axis potential can be uniquely matched to one of the several global solutions from the Bäcklund method.

2.3.3 Riemann-Hilbert problems

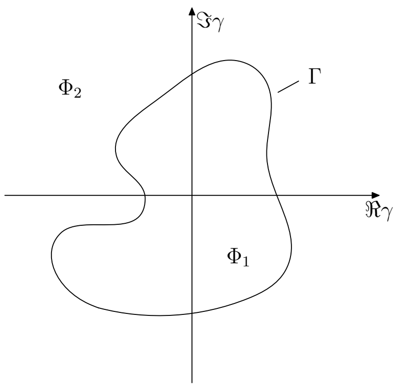

Figure 2.3: The setting of the Riemann Hilbert Problem.

We sketch how Riemann-Hilbert problems can be used to solve the Linear Problem. On the complex λ-plane let a closed contour Γ - which may extend to infinity - be given. Let a matrix valued function $G(\lambda)$ be given. Now one wants to construct a matrix function $\Phi_1(\lambda)$ which is analytic inside Γ and a $\Phi_2(\lambda)$ which is analytic outside Γ, such that

$$\Phi_1(\lambda) \cdot \Phi_2(\lambda) = G(\lambda) \quad |_\Gamma. \tag{2.4}$$

One can also consider the form

$$\tilde{\Phi}_1(\lambda) - \tilde{\Phi}_2(\lambda) = G(\lambda) \quad |_\Gamma, \tag{2.5}$$

because $\Phi_1(\lambda) \cdot \Phi_2(\lambda) \equiv \Phi_1(\lambda) \cdot (\mathbb{1} + \Phi_2(\lambda) - \mathbb{1}) = \Phi_1(\lambda) - \Phi_1(\lambda)(\mathbb{1} - \Phi_2(\lambda))$, one can write $\tilde{\Phi}_1(\lambda) \equiv \Phi_1(\lambda)$ and $\tilde{\Phi}_2(\lambda) \equiv \Phi_1(\lambda)(\mathbb{1} - \Phi_2(\lambda))$. The possible solutions are not unique, but uniqueness can be established if one fixes Φ_1 and Φ_2 at one point in their domain of analyticity. A regular Riemann-Hilbert Problem can be reformulated as an ordinary linear integral equation.

Application to the ISM The idea is to solve the Linear Problem with the help of a Riemann-Hilbert problem for $\mathbf{\Phi}$. Therefore one has to motivate the choice of the curve Γ where $\mathbf{\Phi}$ has a jump. Of course then the Linear Problem has to fail on Γ. This can be motivated physically, as we will explain in the case of the Ernst equation (see following section).

2.4 The Ernst equation and the Inverse Scattering Method

First we want to show how the above mentioned techniques are used in the case of Einstein's equations in the stationary and axisymmetric case, that is the Ernst equation. Then we want to prepare the solution of the Ernst equation for a rotating Black Hole via the Linear Problem in the next chapter.
There are several Linear Problems for the Ernst equation: ([Maison(1978)], [Belinski and Zakharov(1978)], [Harrison(1978)], [Neugebauer(1978)],

[Hauser and Ernst(1979)], [C. Hoenselaers and Xantopolous(1979)]);
we use the one of G. Neugebauer [Neugebauer and Kramer(1983)]:

$$\mathbf{\Phi}_{,z} = \left\{ \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} + \lambda \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \right\} \mathbf{\Phi} \tag{2.6}$$

$$\mathbf{\Phi}_{,\bar{z}} = \left\{ \begin{pmatrix} \bar{A} & 0 \\ 0 & \bar{B} \end{pmatrix} + \frac{1}{\bar{\lambda}} \begin{pmatrix} 0 & \bar{A} \\ \bar{B} & 0 \end{pmatrix} \right\} \mathbf{\Phi}.$$

$\mathbf{\Phi}$ is a complex valued 2×2-matrix, $A(z, \bar{z})$ and $B(z, \bar{z})$ are functions which map $\mathbb{C}^2$ onto $\mathbb{C}$. λ is the *spectral parameter.* In this form it has been used by Meinel and Neugebauer [Neugebauer and Meinel(2003)]. Further details are presented in chapter 3. Using the Linear Problem, Meinel and Neugebauer have obtained several nontrivial solutions to the Ernst equation. The three above mentioned techniques to solve the Linear Problem have been used by them, the integration of the Linear Problem along certain paths, the Polynomial Bäcklund transformation and the Riemann Hilbert problem.

2.4.1 Boundary value problems

An important step is to set boundary conditions to possible solutions. One restricts oneself to a certain class of spacetimes, e.g. stationary and axisymmetric vacuum Black Holes or discs of dust. This leads to constraints to the Ernst potential and thus to the solution of the Linear Problem.

Integration of the Linear Problem with constraints From more general considerations many properties of the Ernst potential can be found for special classes of spacetimes. In turn they can be used as constraints to find solutions which belong to these classes. In some cases - e.g. a rotating Black Hole - the Linear Problem becomes so simple that an analytic solution can be found with moderate effort. The resulting solution then has to be extended to the whole space. Therefore one makes use of the fact that the previously determined asymptotically smooth axis potential determines the Ernst potential everywhere. ([Geroch(1970)], [Hansen(1974)], [Thorne(1980)]) From a C^∞-axis potential one is able to calculate the multipole moments of any order. As the spacetime is asymptotically flat and stationary, the multipole moments uniquely determine the metric tensor of the vacuum spacetime.

Riemann Hilbert Problems and the Ernst equation The Linear Problem fails if its coefficients become singular. Let us consider the case where λ approaches zero or to infinity. There has to be a physical reason for the failure, as the Ernst equation holds everywhere in the vacuum region. As we consider only asymptotically flat spacetimes, a possible physical reason for the failure is that the *surface of a body of fluid or dust is reached.* Then the Ernst equation and the Linear Problem fail and this curve is a natural candidate for the jump curve Γ. It is important to note that the exterior solution found in this way may also be regular beyond this curve. But it is cut in its domain of validity and matched to an interior solution. In the case of a rotating disk of dust the contour Γ is an interval on the ρ-axis. The Ernst potential on the axis can be determined by solving a regular Riemann Hilbert problem. The complete solution is obtained by solving a more general Riemann Hilbert problem.

There is no systematic way known yet how to obtain physically reasonable solutions from the Linear Problem. However some combinations of the above mentioned methods lead to remarkable success. We will give some examples:

2.4.2 Examples of exact solutions obtained by the Linear Problem

Rotating Black Holes - the Kerr solution It is not *a priori* clear in which way the Black Hole solution can be obtained from the Linear Problem. But it turns out that a *twofold Bäcklund transformation* of the trivial solution $f \equiv 1$ (Minkowski space) can be matched with the Ernst potential on the axis. The transformation itself has several solutions, but only one fits to the independently calculated axis value of the Ernst potential f. This solution is unique as its axis potential is the one that has been calculated independently. This *constructive uniqueness proof* was carried out in [Neugebauer and Meinel(2003)]. In chapter 4 it shall be extended to a Black Hole with degenerate horizon.

Other objects The *rotating disk of dust* has already been mentioned. As its extent in ζ-direction is zero the Linear Problem holds everywhere in the (ρ, ζ)-plane exept on an interval on the ρ-axis. It is rather easy to argue that a regular Riemann-Hilbert problem leads to the Ernst potential on the axis [Neugebauer and Meinel(2003)] but it requires more ingenuity to embed it in a more general Riemann-Hilbert problem which then leads to the complete solution. *Multiple Black Holes* can be constructed in a straightforward manner according to the method in chapter 3. From given boundary conditions, the Ernst potential on the axis can in principle be determined. In particular it turns out that a spacetime with two non-degenerate event horizons on the ζ-axis can be generated by a Bäcklund transformation. Although it is widely expected that this *Double Kerr solution* is not regular (not all metric functions have the right properties which might be an indication for the presence of strange matter), a rigorous proof is still missing.

Chapter 3

Solving the Ernst equation via a Linear Problem

We introduce the Linear Problem for the Ernst equation as it is used by Neugebauer und Meinel [Neugebauer and Meinel(2003)]. This section is very technical as many calculations are performed.

One the one hand's side the Ernst equation is a part of Einstein's equations in a stationary and axisymmetric spacetime, as shown in Chapter 1. Due to symmetry the 'rest' of Einstein's equation can automatically be fulfilled. The Ernst equation is nonlinear which is a fundamental property of the equations of General Relativity.[1]
On the other hand's side it is an alternative formulation of the integrability condition of a linear matrix problem. If the Ernst equation holds, this Linear Problem can be solved by line integration. This provides an elegant way to solve the Ernst equation via a linear boundary value problem, which is demonstrated in the course of this chapter. We focus on a stationary and axisymmetric vacuum spacetime which contains a Black Hole with a single horizon. It is shown how physically motivated boundary conditions lead to a solution of the Linear Problem on the axis of symmetry. After a longer calculation we will arrive at the Ernst potential on the ζ-axis. In a second step which is not presented here [Neugebauer and Meinel(2003)] a number of global solutions will be obtained by a Bäcklund transform, and one of them will fit to the axis potential. Finally the Kerr Black Hole results, excluding the extreme case.

The last section (3.4) deals with coordinate independent formulations of both the Ernst equation and the Linear Problem. Thus Einstein's equations in an s&a vacuum spacetime $\mathcal{M}$ can be solved everywhere on $\mathcal{M}$ with the help of the Linear Problem. Due to the choice of Weyl coordinates the singularity of the Kerr metric is not covered by this system. Any further 'singularity' which occurs in the context of the Linear Problem in Weyl coordinates will be a coordinate singularity. Thus we make sure that our attempt of the uniqueness proof in Weyl coordinates is physically reasonable and the singularities that occur are due to a defect of these coordinates on the degenerate horizon and not due to physical reasons.

[1] A rotating Black Hole spacetime cannot be obtained by transforming the Schwarzschild solution into a rotating frame!

3.1 The Linear Problem for the Ernst equation

The subsequent three sections are very close to the review article of Neugebauer and Meinel [Neugebauer and Meinel(2003)]. However many of the details that have been omitted in this article are presented here.

3.1.1 The Linear Problem

The following linear matrix problem is associated with the Ernst equation:

$$\begin{aligned}\mathbf{\Phi},_z &= \left\{\begin{pmatrix} B & 0 \\ 0 & A\end{pmatrix} + \lambda\begin{pmatrix} 0 & B \\ A & 0\end{pmatrix}\right\}\mathbf{\Phi} \\ \mathbf{\Phi},_{\bar z} &= \left\{\begin{pmatrix} \bar A & 0 \\ 0 & \bar B\end{pmatrix} + \frac{1}{\bar\lambda}\begin{pmatrix} 0 & \bar A \\ \bar B & 0\end{pmatrix}\right\}\mathbf{\Phi}.\end{aligned} \tag{3.1}$$

Here the complex variable $z := \rho + i\zeta$ has been introduced. $\mathbf{\Phi}$ depends on z, $\bar z$ and λ and it is a complex-valued 2×2-matrix. $A(z,\bar z)$ and $B(z,\bar z)$ are functions which map $\mathbb{C}^2$ onto $\mathbb{C}$. λ is a further variable and is called *spectral parameter*. It plays an important role. The idea of the Inverse Scattering method (the Linear Problem) is to consider $\mathbf{\Phi}$ for fixed, but arbitrary $(z,\bar z)$ as holomorphic function in λ and to calculate A, B and f afterwards.

For some purposes it is useful to introduce a further parameter K:

$$\lambda(z,\bar z,K) = \sqrt{\frac{K - i\bar z}{K + iz}},$$

from which one obtains

$$\lambda,_z = \frac{\lambda}{4\rho}(\lambda^2 - 1); \qquad \lambda,_{\bar z} = \frac{1}{4\rho\lambda}(\lambda^2 - 1).$$

A and B do not depend on K.
Sometimes we will use the common abbreviation

$$\mathbf{\Phi},_z = \boldsymbol{U}\mathbf{\Phi}; \qquad \mathbf{\Phi},_{\bar z}\, \boldsymbol{V}\mathbf{\Phi}$$

for the Linear Problem. One is interested in $\mathbf{C}^2$-solutions which fulfil the *integrability condition*

$$\mathbf{\Phi},_{z\bar z} = \mathbf{\Phi},_{\bar z z}\,.$$

In this case, the problem can be solved by line integration, which is of big importance for the Inverse Scattering Method. In terms of $\boldsymbol{U}$ and $\boldsymbol{V}$, the integrability condition reads

$$\mathrm{UV} + \mathrm{U},_{\bar z} = \mathrm{VU} + \mathrm{V},_z\,.$$

Now we can express the integrability condition in terms of A and B and their derivatives, just by calculating the second order partial derivatives of $\mathbf{\Phi}$. The result is a 2×2 matrix equation which can be regarded as a system of four complex equations. They must hold for any $\lambda \in \mathbb{C}$. After eliminating $\bar{A}_{,z}$, $\bar{B}_{,z}$ and the derivatives of λ we arrive at

$$A_{,\bar{z}} = A(\bar{B} - \bar{A}) - \frac{1}{4\rho}(A + \bar{B}); \qquad B_{,\bar{z}} = B(\bar{A} - \bar{B}) - \frac{1}{4\rho}(B + \bar{A}).$$

This is a system of first order differential equations for A and B. Now we introduce another complex function f and *choose* for A and B:

$$A = \frac{f_{,z}}{f + \bar{f}}; \qquad B = \frac{\bar{f}_{,z}}{f + \bar{f}}\ , \tag{3.2}$$

hence

$$\bar{A} = \frac{\overline{f_{,z}}}{f + \bar{f}}; \qquad \bar{B} = \frac{\overline{\bar{f}_{,z}}}{f + \bar{f}}\ .$$

By resubstitution in the system and after rearrangement one gets:
Equation from A, z:

$$2\Re f(f_{,z\bar{z}} + \frac{1}{4\rho}(f_{,z} + \overline{\bar{f}_{,z}})) = f_{,z}\,(f_{,\bar{z}} + \bar{f}_{,\bar{z}} + \overline{\bar{f}_{,z}} - \overline{f_{,z}}), \tag{3.3}$$

equation from B, z:

$$2\Re f(f_{,z\bar{z}} + \frac{1}{4\rho}(\overline{f_{,z}} + \bar{f}_{,z}\,)) = \bar{f}_{,z}\,(f_{,\bar{z}} + \bar{f}_{,\bar{z}} + \overline{f_{,z}} - \overline{\bar{f}_{,z}}). \tag{3.4}$$

Now we introduce new variables, ρ and ζ:

$$z = \rho + i\zeta; \qquad \bar{z} = \rho - i\zeta \qquad \Longleftrightarrow \qquad \rho = \frac{z + \bar{z}}{2}; \qquad \zeta = \frac{z - \bar{z}}{2i}.$$

One can express the derivatives by z and $\bar{z}$ through those by ρ and ζ.

$$\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial \rho} - \frac{i}{2}\frac{\partial}{\partial \zeta}; \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial \rho} + \frac{i}{2}\frac{\partial}{\partial \zeta}.$$

This replacement of variables has nothing to do with complex analysis. Because the map $z \longrightarrow \bar{z}$ is not holomorphic, they both are indeed two independent variables. It will turn out that the two equations above are complex conjugate to each other. For $f = u + iv$ we find for the derivatives

$$\begin{aligned}
f, z &= \frac{1}{2}u_{,\rho} + \frac{i}{2}v_{,\rho} - \frac{i}{2}u_{,\zeta} + \frac{1}{2}v_{,\zeta} &&\Longrightarrow\ \overline{f_{,z}} = ... \\
\bar{f}_{,z} &= \frac{1}{2}u_{,\rho} - \frac{i}{2}v_{,\rho} - \frac{i}{2}u_{,\zeta} - \frac{1}{2}v_{,\zeta} &&\Longrightarrow\ \overline{\bar{f}_{,z}} = ... \\
f_{,\bar{z}} &= \frac{1}{2}u_{,\rho} + \frac{i}{2}v_{,\rho} + \frac{i}{2}u_{,\zeta} - \frac{1}{2}v_{,\zeta} &&\Longrightarrow\ \overline{f_{,\bar{z}}} = ... \\
\bar{f}_{,\bar{z}} &= \frac{1}{2}u_{,\rho} - \frac{i}{2}v_{,\rho} + \frac{i}{2}u_{,\zeta} + \frac{1}{2}v_{,\zeta} &&\Longrightarrow\ \overline{\bar{f}_{,\bar{z}}} = ...
\end{aligned} \tag{3.5}$$

Now it is clear that

$$\overline{f_{,z}} \neq \bar{f}_{,z} \quad \text{but that} \quad \overline{f_{,\bar{z}}} = \bar{f}_{,z} \quad \textit{and} \quad \overline{f_{,z}} = \bar{f}_{,\bar{z}}\,.$$

With the help of this array of equations one can reformulate the expressions in equations (3.3) and (3.4). What one finds is that equation (3.3) is the complex conjugate of equation (3.4). Writing $f_{,z\bar{z}}$ in terms of ρ and ζ one arrives at

$$\begin{aligned} 2\Re f\left(\frac{1}{4}(f_{,\rho\rho}+f_{,\zeta\zeta})+\frac{1}{4\rho}f_{,\rho}\right) &= \frac{1}{2}(u_{,\rho}+iv_{,\rho}-iu_{,\rho}+v_{,\rho})\cdot(u_{,\rho}+iv_{,\rho}+iu_{,\zeta}-v_{,\zeta}) \\ &= \frac{1}{2}(f_{,\rho}^{2}+f_{,\zeta}^{2}), \end{aligned}$$

which is the *Ernst equation in Weyl coordinates*:

$$\Re f(f_{,\rho\rho}+f_{,\zeta\zeta}+\frac{1}{\rho}f_{,\rho}) = f_{,\rho}^{2}+f_{,\zeta}^{2}. \tag{3.6}$$

3.1.2 Towards the solution of the Linear Problem on the axis and the horizon of a stationary and axisymmetric spacetime

One can make further assumptions on the structure of $\mathbf{\Phi}$ without loss of generality. (No proofs of the results of this subsection are provided here.)
$\mathbf{\Phi}$ always can be chosen to look like

$$\mathbf{\Phi} = \begin{pmatrix} \psi(\rho,\zeta,\lambda) & \psi(\rho,\zeta,-\lambda) \\ \chi(\rho,\zeta,\lambda) & -\chi(\rho,\zeta,-\lambda) \end{pmatrix}, \tag{3.7}$$

with complex C^2-functions ψ and χ. Note that both columms of $\mathbf{\Phi}$ are independent solutions of the Linear Problem. Moreover one can replace ψ by χ and vice versa:

$$\bar{\psi}(\rho,\zeta,\frac{1}{\bar{\lambda}}) = \chi(\rho,\zeta,\lambda).$$

Normalisation of ψ and χ: For $K \longrightarrow \infty$, $\lambda = -1$ one can fix ψ and χ:

$$\psi(\rho,\zeta,-1) = \chi(\rho,\zeta,-1) = 1. \tag{3.8}$$

For the other branch of the Riemann surface ($\lambda = +1$) one can calculate the behaviour from the Linear Problem:

$$\frac{\psi(\rho,\zeta,1)_{,z}}{\Re\psi(\rho,\zeta,1)} = \frac{\bar{f}(\rho,\zeta)_{,z}}{\Re f(\rho,\zeta)} \qquad \frac{\chi(\rho,\zeta,1)_{,z}}{\Re\chi(\rho,\zeta,1)} = \frac{f(\rho,\zeta)_{,z}}{\Re f(\rho,\zeta)}.$$

These equations allow any solution of the form

$$\psi(\rho,\zeta,1) = a\bar{f}+ib, \qquad \chi(\rho,\zeta,1) = cf+id$$

with $a,b,c,d \in \mathbb{R}$. They are chosen in a way that

$$\psi(\rho,\zeta,1) = \bar{f}(\rho,\zeta), \qquad \chi(\rho,\zeta,1) = f(\rho,\zeta). \tag{3.9}$$

The corotating Linear Problem The metric of a stationary and axisymmetric spacetime retains its form if one transforms to a corotating system by

$$t' = t \qquad \phi' = \phi - \omega t; \qquad \Longrightarrow \qquad \xi'^i = \xi^i + \omega\eta^i \qquad \eta'^i = \eta^i.^{(2)} \tag{3.10}$$

The line element

$$ds^2 = e^{2U}(e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\phi^2) - e^{2U}(dt + a d\phi)^2$$

keeps its form under the transformation (3.10). One can rearrange the terms to new metric functions $\tilde{U}$, $\tilde{a}$, $\tilde{k}$:

$$ds^2 = e^{2\tilde{U}}(e^{2\tilde{k}}(d\rho^2 + d\zeta^2) + \rho^2 d\phi^2) - e^{2\tilde{U}}(dt + \tilde{a} d\phi)^2$$

where the tilded functions are given as follows:

$$\begin{aligned}
e^{2\tilde{U}} &= e^{2U} + 2a \cdot e^{2U}\omega + (e^{2U} \cdot a^2 - \rho^2 \cdot e^{-4U})\omega^2 && (3.11)\\
\tilde{a} &= (a \cdot e^{2U} + a^2 \cdot e^{2U}\omega - \rho^2 \cdot e^{-2U}\omega)e^{-2\tilde{U}} && (3.12)\\
&= \frac{1}{\omega}\Big(1 - \frac{(1 + a\omega)e^{2U}}{e^{2U} + 2a \cdot e^{2U}\omega + (e^{2U} \cdot a^2 - \rho^2 \cdot e^{-2U})\omega^2}\Big)\\
&= \frac{1}{\omega}\Big(1 - \frac{(1 + a\omega)}{1 + 2a \cdot \omega + (a^2 - \rho^2 \cdot e^{-4U})\omega^2}\Big) && (3.13)\\
e^{2\tilde{k}} &= e^{2(k-U)} \cdot e^{2\tilde{U}}\\
&= e^{2k}(1 + 2a \cdot \omega + (a^2 - \rho^2 \cdot e^{-4U})\omega^2).
\end{aligned}$$

$$(3.14)$$

The Ernst equation remains in the same form, but now it is an equation for $\tilde{f}$:

$$(\Re\tilde{f})\triangle\tilde{f} = (\nabla\tilde{f})^2, \qquad \text{with } \tilde{f} = e^{2\tilde{U}} + i\tilde{b},$$

where $\tilde{b}$ is related to $\tilde{a}$ by

$$\tilde{b}_{,\rho} = -\frac{1}{\rho}e^{4\tilde{U}}\tilde{a}_{,\zeta}\,; \qquad \tilde{b}_{,\zeta} = \frac{1}{\rho}e^{4\tilde{U}}\tilde{a}_{,\rho}\,.$$

The Linear Problem also remains form invariant, with new functions $\widetilde{\boldsymbol{\Phi}}$, $\tilde{A}$ and $\tilde{B}$. One finds that [Neugebauer and Meinel(2003)]

$$\widetilde{\boldsymbol{\Phi}} = \boldsymbol{R}\boldsymbol{\Phi},$$

where

$$\boldsymbol{R} = \left\{\begin{pmatrix} 1 + a\omega - \rho e^{-2U}\omega & 0 \\ 0 & 1 + a\omega + \rho e^{-2U} \end{pmatrix} + i(K + iz)\omega e^{-2U}\begin{pmatrix} -1 & -\lambda \\ \lambda & 1 \end{pmatrix}\right\}.$$

This for example can be proven in a straight forward calculation with the help of a computer.

It has become clear how the Ernst potential is related to the given Linear Problem. The next step will be that we integrate the Linear Problem along the axis of symmetry, which can be carried out with moderate effort. The result is valid for any s&a asymptotically flat vacuum spacetime. In addition there is also a Linear Problem in the co-rotating frame. This will also be evaluated, as in this system the boundary condition for a Black Hole horizon can be expressed in a simple way.

[2] The change of sign in the transformation formula of the vectors occurs because the coordinates transform like components of forms (covectors), namely covariant.

3.2 The integration of the Linear Problem

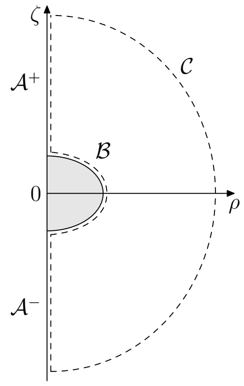

Figure 3.1: The path of integration in the (ρ-ζ)-plain.

Now the task is to find a sufficiently general solution of the Linear Problem on a certain path. Due to the integrability conditions one has the choice of a path whichis convenient to handle. This is of course the axis of symmetry, where $\rho = 0$ and thus $\lambda = \pm 1$, see figure 3.1. As we only deal with asymptotic vacuum solutions (for sufficiently large ρ and ζ there stress-energy tensor vanishes), this integration has to be carried out only once and will be valid for any isolated object. Somewhere on the axis, for convenience symmetric to $\zeta = 0$, the 'body' is situated, that is the object that shall be modeled. Both parts of the axis of symmetry lead to independent fundamental systems for $\mathbf{\Phi}$, which have to be interconnected on the path $\mathcal{C}$ at infinity. On the surface $\mathcal{B}$ of the body - this may be the surface of a star or a Black Hole horizon - there are physical constraints whichyield a special solution. The shape of the body and the constraints can be various, which makes it impossible in the general case to give an analytic solution.
In the following section we give the details of the calculation for one Black Hole horizon as it is shown in [Neugebauer and Meinel(2003)]. The constraints are illustrated in figure 3.2 and are explained in section 3.2.2. This is the simplest possible case apart from the trivial solution, the Minkowski space.

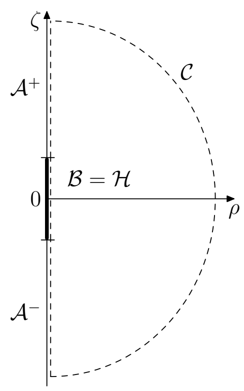

Figure 3.2: The path of integration for a Black Hole.

3.2.1 Integration of the Linear Problem along the axis $\mathcal{A}^+\mathcal{C}\mathcal{A}^-$

On the axis one only has to consider the ζ-derivative. One can rewrite the problem and replace A and B by (3.2). The result is

$$\mathbf{\Phi}_{,\zeta} = \frac{1}{f+\bar{f}} \begin{pmatrix} \bar{f}_{,\zeta} & \bar{f}_{\zeta} \\ f_{,\zeta} & f_{,\zeta} \end{pmatrix} \mathbf{\Phi}; \qquad \text{for } \lambda = 1.$$

One finds two linearly independent solutions that establish a fundamental system.

$$L_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \qquad L_2 = \begin{pmatrix} \bar{f} \\ f \end{pmatrix}.$$

The general solution contains four unspecified functions of K,

$$\mathbf{\Phi} = \begin{pmatrix} \bar{f} & 1 \\ f & -1 \end{pmatrix} \begin{pmatrix} F(K) & H(K) \\ G(K) & I(K) \end{pmatrix}.$$

This solution consists of the ζ-depending fundamental system and of K-depending 'integration constants'. We *choose* the following initial values: On $\mathcal{A}^+$, $\lambda = -1$, ψ and χ are set equal to 1 for a ζ_0. Writing down $\mathbf{\Phi}(\zeta_0, K)$, we see that the right column can only be $(1,-1)^T$ if H is zero and I is equal to 1 for any K. The solution on $\mathcal{A}^+$ then reads

$$\mathcal{A}^+) \qquad \mathbf{\Phi}(\zeta) = \begin{pmatrix} \bar{f}(\zeta) & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} F(K) & 0 \\ G(K) & 1 \end{pmatrix}. \tag{3.15}$$

Interconnection with $\mathcal{A}^-$ via spatial infinity (curve $\mathcal{C}$) For $\zeta \longrightarrow +\infty$, A and B tend to zero and $\mathbf{\Phi}$ becomes a function of K, or λ only. If one introduces a parametrization of ρ and ζ given by

$$\zeta = r \cdot \cos(\phi), \qquad \rho = r \cdot \sin(\phi).$$

λ can be written as

$$\lambda = \sqrt{\frac{K - \zeta - i\rho}{K - \zeta + i\rho}} = \sqrt{\frac{K - r \times e^{i\phi}}{K - r \times e^{-i\phi}}} \xrightarrow{\frac{K}{r} \approx \frac{K}{\zeta} \longrightarrow 0} \sqrt{e^{2i\phi}} \stackrel{*}{=} e^{i\phi}.$$

The last equality sign refers to $\lambda = +1$ for $\zeta \longrightarrow +\infty$. Now, if ϕ goes from 0 to π, $\lambda = 1$ turns into $\lambda = -1$. Thus along $\mathcal{C}$ the sheets of the Riemannian surface of the function λ are exchanged.
Now one makes a similar ansatz for the general solution on $\mathcal{A}^-$, with new functions M, N, O and P of K:

$$\mathbf{\Phi} = \begin{pmatrix} \bar{f}(\zeta) & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} M(K) & N(K) \\ P(K) & Q(K) \end{pmatrix} = \begin{pmatrix} \bar{f}M(K) + P(K) & \bar{f}N(K) + Q(K) \\ fM(K) - P(K) & fN(K) - Q(K) \end{pmatrix}$$

If one travels along $\mathcal{C}$, the components of $\mathbf{\Phi}$ do not change. Hence we can compare the two solutions for $\mathcal{A}^+$ and $\mathcal{A}^-$. But one has to respect two properties of $\mathbf{\Phi}$: As $\lambda \longrightarrow -\lambda$ on $\mathcal{C}$, one has to compare the right column of $\mathbf{\Phi}|_{\mathcal{A}^+}$ (for $\lambda = -1$) with the left column of $\mathbf{\Phi}|_{\mathcal{A}^-}$ (for $\lambda = 1$) and vice versa. And one further has to keep in mind that $\mathbf{\Phi}_{22}$ and $\mathbf{\Phi}_{21}$ cange their sign as χ remains unchanged. For large values of r, f tends to 1. In the limit of a very large r, we now can compare the two solutions for $\mathcal{A}^+$ and $\mathcal{A}^-$:

$$\begin{pmatrix} F(K)+G(K) & 1 \\ F(K)-G(K) & -1 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} N(K)+Q(K) & M(K)+P(K) \\ -(N(K)-Q(K)) & -(M(K)-P(K)) \end{pmatrix},$$

whence

$$P(K)\equiv 0; \quad M(K)\equiv 1 \quad Q(K)\equiv F(K) \quad N(K)\equiv G(K).$$

The general solution is now

$$\mathcal{A}^-) \qquad \mathbf{\Phi}(\zeta,K) = \begin{pmatrix} \bar{f}(\zeta), & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} 1 & G(K) \\ 0 & F(K) \end{pmatrix}. \tag{3.16}$$

This prodcedure is discussed in detail in [Meinel and Neugebauer(1995)]. For solving the boundary value problem it is helpful to consider the Linear Problem in the corotating system as well. On the axis the metric function a is equal to zero and ρ is zero anyway, and one can write down the co-rotating general solutions

$$\mathcal{A}^+) \ \widetilde{\mathbf{\Phi}} = \left[\mathbb{1}_2 + i(K-\zeta)\omega e^{-2U}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} \bar{f}(\zeta) & 1 \\ f(\zeta) & -1 \end{pmatrix}\begin{pmatrix} F(K) & 0 \\ G(K) & 1 \end{pmatrix}\right], \tag{3.17}$$

$$\mathcal{A}^-) \ \widetilde{\mathbf{\Phi}} = \left[\mathbb{1}_2 + i(K-\zeta)\omega e^{-2U}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} \bar{f}(\zeta) & 1 \\ f(\zeta) & -1 \end{pmatrix}\begin{pmatrix} 1 & G(K) \\ 0 & F(K) \end{pmatrix}\right]. \tag{3.18}$$

$\mathbf{\Phi}$ is a unique function of λ but a twofold one of K. At the branch points $K_B=\zeta$ however, λ is unique and thus are ψ and χ. Hence there exists an analytical continuation of the Ernst potential on the axis from which $F(K)$ and $G(K)$ can be calculated. Vice versa, $f(\zeta)$ follows from F and G for $K=\zeta$:

$$\mathcal{A}^+, (K_b=\zeta): \qquad \mathbf{\Phi}(\zeta) = \begin{pmatrix} \psi & \psi \\ \chi & -\chi) \end{pmatrix} = \begin{pmatrix} \bar{f}(\zeta), & 1 \\ f(\zeta) & -1 \end{pmatrix}\begin{pmatrix} F(\zeta) & 0 \\ G(\zeta) & 1 \end{pmatrix}, \tag{3.19}$$

$$\mathcal{A}^-, (K_B=\zeta): \qquad \mathbf{\Phi}(\zeta) = \begin{pmatrix} \psi & \psi \\ \chi & -\chi) \end{pmatrix} = \begin{pmatrix} \bar{f}(\zeta), & 1 \\ f(\zeta) & -1 \end{pmatrix}\begin{pmatrix} 1 & G(\zeta) \\ 0 & F(\zeta) \end{pmatrix}, \tag{3.20}$$

whence

$$\mathcal{A}^+: \qquad F(\zeta) = \frac{2}{f(\zeta)+\overline{f(\zeta)}}, \qquad G(\zeta) = \frac{f(\zeta)-\overline{f(\zeta)}}{f(\zeta)+\overline{f(\zeta)}}, \tag{3.21}$$

$$\mathcal{A}^-: \qquad F(\zeta) = \frac{2f(\zeta)\overline{f(\zeta)}}{f(\zeta)+\overline{f(\zeta)}}, \qquad G(\zeta) = \frac{\overline{f(\zeta)}-f(\zeta)}{f(\zeta)+\overline{f(\zeta)}}. \tag{3.22}$$

Conclusion On the regular parts of the axis $\mathbf{\Phi}$ and $\widetilde{\mathbf{\Phi}}$ can explicitly be expressed by the Ernst potential on the axis $f(\zeta)$ and its analytic continuations $F(K)$ and $G(K)$. The crucial point is the integration along $\mathcal{B}$, the surface of the considered object. Here, the physical nature of the problem - apart from symmetries - is specified. In some situations it is possible to find analytic solutions for F and G. One example is a rotating Black Hole spacetime and another one is a rigidly rotating disk of dust.

3.2.2 The Linear Problem on the surface of a Black Hole

As we consider asymotically flat spacetimes, we can always use the general solution along $\mathcal{A}^+\mathcal{C}\mathcal{A}^-$, which has to be matched to a solution on $\mathcal{B}$ which satisfies physical boundary conditions. After the choice of the body (a model of an astrophsical or abstract object) one has to look for constraints of the metric functions U, a, k and $\tilde{U}, \tilde{a}, \tilde{k}$ on its surface. Then one has calculate a sufficiently general solution on $\mathcal{B}$ with respect to the valid constraints. The solution on $\mathcal{B}$ is then matched to the general solution on $\mathcal{A}^+\mathcal{C}\mathcal{A}^-$ by assuming that $\mathbf{\Phi}$ is *continous* at the meeting points of $\mathcal{B}$ with the axis. This brings about some problems as in some cases there is no continuity. As we will see, in the case of an extreme Kerr Black Hole the horizon is hidden within a point and thus $\mathbf{\Phi}$ does not exist on $\mathcal{H}$ in Weyl coordinates.
It was shown by Hu Hesheng ([Hesheng(1990)] that the Linear Problem can be written in terms of differential forms. It holds in any *valid* coordinate system. If the coordinate description of the manifold turns pathologic at some region, then of course there will be serious problems if one is restricted to special coordinates. When coordinates are chosen in a way that they reflect symmetries, problems occur when the physical interpretation does not accord with mathematical properties anymore. An example is the Killing vector ξ in a rotating Black Hole spacetime. Normally it is related to time translation symmetry. But this can only be done, if $(\xi|\xi)$ is timelike. However ξ can become null or spacelike if a *horizon* or an *ergosurface* occurs in the spacetime. Unfortunately on these surfaces the metric becomes singular when it is written down in symmetry-adapted coordinates. These so called coordinate singularities are due to the choice of the coordinate system and that they can be removed by the choice of a valid coordinate system. The do not question the validity of General Relativity at these places.
In the uniqueness theory of Black Holes one encounters a further difficulty: One cannot discuss the solutions which first have to be found. One rather has to make reasonable assumptions on boundary conditions from which one can try to deduce unique solutions. The 'surface' of a Black Hole is such a place. We already know the Kerr solution and can wonder wether one should consider the horizon or the ergosurface as the 'surface of the body' which enters into the Linear Problem. However this is not necessary. As the Inverse Scattering Method is purely mathematical, one can choose the simplest 'surface'! Without any doubt this is the horizon, as it lies on the axis:

$$\mathcal{H}: \qquad \rho = 0; \quad K_1 \geq \zeta \geq K_2; \qquad e^{2\tilde{U}} = 0; \qquad \Longrightarrow a = \frac{1}{\Omega_H} = \mathrm{const}|_{\mathcal{H}}.$$

The transfer matrix $\boldsymbol{R}$ under these conditions reads ($\lambda = 1$)

$$\boldsymbol{R} = i(K - \zeta)\Omega_H e^{-2U} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Following the procedure of finding $\mathbf{\Phi}$ on the axis once again we find an ansatz with unspecified functions U, V, W, X of K:

$$\mathcal{B} = \mathcal{H}: \quad \mathbf{\Phi}(\zeta, K) = \begin{pmatrix} \bar{f}(\zeta), & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} U(K) & V(K) \\ W(K) & X(K) \end{pmatrix}, \tag{3.23}$$

$$\mathcal{B} = \mathcal{H}: \quad \widetilde{\mathbf{\Phi}}(\zeta, K) = 2i\Omega_H(K - \zeta) \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U(K) & V(K) \\ W(K) & X(K) \end{pmatrix}. \tag{3.24}$$

3.3 The Ernst potential everywhere on the axis

3.3.1 Matching the solutions

We have derived several solutions of the Linear Problem which are valid on a certain interval only. To get a unique solution everywhere on the ($\rho = 0$) - curve the imcomplete solutions have to be matched. Therefore we assume that in the matching points K_1 and K_2, $\mathbf{\Phi}$ and $\widetilde{\mathbf{\Phi}}$ are both continous (for detailed arguments, cf. 4.3).
We have four matrix equations, two for K_1 and K_2 and two for $\mathbf{\Phi}$ and $\widetilde{\mathbf{\Phi}}$. The aim is to eliminate the $\begin{pmatrix} U(K) & V(K) \\ W(K) & X(K) \end{pmatrix}$-matrix. The system is not overdetermined as each of the four matrix equations contains a matrix prefactor of determinant 0. Thus one has to consider linear combinations of the equations in order to eliminate the prefactor. One possible reduction is the following system of two matrix equations:

$$\begin{bmatrix} f_1 & -1 \\ f_1 + 2i\Omega_H(K-K_1) & -1 \end{bmatrix} \begin{bmatrix} F(K) & 0 \\ G(K) & 1 \end{bmatrix} = \begin{bmatrix} f_1 & -1 \\ 2i\Omega_H(K-K_1) & 0 \end{bmatrix} \begin{bmatrix} U(K) & V(K) \\ W(K) & X(K) \end{bmatrix},$$

$$\begin{bmatrix} f_2 & -1 \\ f_2 + 2i\Omega_H(K-K_2) & -1 \end{bmatrix} \begin{bmatrix} 1 & G(K) \\ 0 & F(K) \end{bmatrix} = \begin{bmatrix} f_2 & -1 \\ 2i\Omega_H(K-K_2) & 0 \end{bmatrix} \begin{bmatrix} U(K) & V(K) \\ W(K) & X(K) \end{bmatrix},$$

where $f_1 = f(\zeta = K_1)$ and $f_2 = f(\zeta = K_2)$. f_1 and f_2 are imaginary: On the horizon $e^{2\tilde{U}} = 0$ and $\rho = 0$. While approaching the axis, a discontinuity occurs, as $a_{\text{axis}} = 0$, but $a_{\mathcal{H}} = \text{const} \neq 0$. The Ernst potential f is not affected by this defect, which leads to the condition $e^{2U}|_{\mathcal{H}\cap\text{axis}} = 0$. This means that the horizon and the ergosurface lead in the same direction into the axis. At the matching points, f is imaginary.
The next step is the elimination of the $(U, V; W, X)$ matrix. This can easily be done because the matrices left of $(U, V; W, X)$ are invertible. We introduce the abbreviations

$$\alpha_1 = 2i\Omega_H(K-K_1); \qquad \alpha_2 = 2i\Omega_H(K-K_2).$$

The resulting matrix equation

$$\begin{pmatrix} (1+\frac{f_1}{\alpha_1})F - \frac{G}{\alpha_1} & -\frac{1}{\alpha_1} \\ \frac{f_1^2}{\alpha_1}F + (1-\frac{f_1}{\alpha_1})G & 1-\frac{f_1}{\alpha_1} \end{pmatrix} = \begin{pmatrix} (1+\frac{f_2}{\alpha_2}) & (1+\frac{f_2}{\alpha_2})G - \frac{F}{\alpha_2} \\ \frac{f_2^2}{\alpha_1} & \frac{f_2^2}{\alpha_2}G + (1-\frac{f_2}{\alpha_2})F \end{pmatrix} \tag{3.25}$$

consists of four scalar equations which have to be satisfied *for any* K. One can solve two of these equations for F and G, which turn out to be rational functions in K. Then we replace them both in the remaining two equations. This leads to a first order polynomial in K. The coefficients have to vanish, whence

$$f_1^2 = f_2^2; \qquad \Omega_H = \frac{i(1+f_1^2)(f_2-f_1)}{2(1-f_1^2)(K_2-K_1)}. \tag{3.26}$$

If $f_1 = f_2$, Ω_H would be zero, which contradicts our assumptions as it leads back to the nonrotating case. The constraints read then

$$f_1 = -f_2; \qquad \Omega_H = \frac{if_1(1+f_1^2)}{(1-f_1^2)(K_1-K_2)}. \tag{3.27}$$

We set the horizon on a symmetric position and choose $K_1 = -K_2$. The constraints simplify the expressions for F and G:

$$F(K) = \frac{4\Omega_H^2(K^2-K_1^2)+4i\Omega_H f_1 K - 2f_1^2}{4\Omega_H^2(K^2-K_1^2)}, \qquad G(K) = \frac{4i\Omega_H K_1 + 2f_1}{4\Omega_H^2(K^2-K_1^2)}. \tag{3.28}$$

Now we are able to calculate f by setting $K = \zeta$. On $\mathcal{A}^+$ we find from (3.21)

$$\mathcal{A}^+) \qquad f(\zeta) = \frac{G+1}{F}|_{K=\zeta} \quad = \quad \frac{\zeta(1+f_1^2)+(f_1^2-1+2f_1)K_1}{\zeta(1+f_1^2)+(1-f_1^2+2f_1)K_1}. \tag{3.29}$$

This is the Ernst potential on the axis for positive ζ. The result for negative ζ is obtained by 3.22. (Here, $f = \frac{F}{G+1}$.) The integration of the Linear Problem on the axis is now complete.

3.3.2 Asymptotic behaviour of f

Interpretation of the solution Of course we still have to relate the parameters f_1 and K_1 to more common physical quantities. This is done by the calculation of the multipole moments of f which is a Taylor expansion in ζ-direction. Once we have calculated them, we can interpret the potential on the axis and can compare it to known solutions of the Ernst equation.

Newtonian limit In the case of asymptotic flatness one can compare the asymptotic form of the metric on the ζ-axis with the one of the Newtonian limit,

$$g_{ab} \cong \eta_{ab} - 2\phi = -1 - 2(-\frac{M}{r}) \quad \Longrightarrow \quad g_{tt} = -e^{2U} \cong -1 - 2U \cong -1 + 2\frac{M}{r}.$$

One can see that far away from the horizon U can be considered as gravitational potential. From a Taylor expansion of f in ζ one finds that

$$M = \frac{1-f_1^2}{1+f_1^2}K_1.$$

M is the total mass of the spacetime which is well defined for a stationary asymptotically flat spacetime. As we solved the vacuum equations, M must lie within the horizon.

The angular momentum The asymptotic expansion of b is

$$b \cong 0 + 0 - 4if_1K_1^2\frac{1-f_1^2}{(1+f_1^2)^2}\frac{1}{\zeta^2} = -4M\frac{if_1K_1}{1+f_1^2}\frac{1}{\zeta^2}.$$

We assume that b contains information about the angular momentum. If we compare f with the Kerr solution, we also can expand b to

$$b \cong \frac{-2J}{\rho^2} \quad \Longrightarrow \quad J = 2M\frac{if_1K_1}{1+f_1^2}.$$

A stronger evidence can be provided if we use the fact that the $\frac{1}{\rho}$-term of $g_{t\phi}$ indicates the presence of angular momentum. a denotes the angular momentum per mass and is a common abbreviation:

$$a := \frac{J}{M} = \frac{2if_1K_1}{1+f_1^2}.$$

The final result is the Ernst potential of rotating Black Hole on the axis in terms of its mass and angular momentum.

$$f(\zeta) = \frac{\zeta - M - i\frac{J}{M}}{\zeta + M - i\frac{J}{M}} = 1 - \frac{4M}{2\zeta + 2M - 2i\frac{J}{M}} \tag{3.30}$$

This is the Ernst potential of the Kerr metric on the axis. By the help of the Linear Problem Meinel and Neugebauer were able to show that the only physically reasonable stationarily rotating vacuum Black Hole spacetimes with non-degenerate horizon are the family of Kerr Black Holes with an angular momentum $0 \neq J < M^2$. If we calculate the axis potential for the case $J \longrightarrow M^2$ by applying the rule of de l'Hospital to the final result, we find that in this case K must turn to zero, or in other words that a horizon that is situated at $\zeta = 0$ is related to the extreme Kerr Black Hole.
This Black Hole is the one with maximal angular momentum, $J = \pm M^2$. It is not included in this proof, as many formulas become senseless for $J \Longrightarrow M^2$. In between it was assumed that the horizon is a non-empty intervall on the ζ-axis. In chapter 4 we will deal with the extreme Kerr Black Hole and the proof of this chapter will be extended to the case of the degenerate horizon.

3.3.3 The Ernst potential everywhere

The Ernst potential on the axis uniquely determines the Ernst potential everywhere. (See chapter 2). Now the question arises how the Ernst potential on the (ρ, ζ)-plain can be determined. As already mentioned, it turned out that the polynomial Bäcklund transform yields a solution that can be matched to the given axis potential.
In [Neugebauer and Meinel(2003)] the transformation looks a bit different than the one in chapter 2:

$$\widetilde{\boldsymbol{\Phi}} = \boldsymbol{\Phi}\boldsymbol{T},$$

where

$$\boldsymbol{T}(K) = \frac{K^2 + \alpha^2 - M^2}{K((K+M)^2 + \alpha^2)} \cdot \begin{pmatrix} K+M & \mathrm{i}\alpha \\ \mathrm{i}\alpha & K+M \end{pmatrix}.$$

One then concludes that $\widetilde{\boldsymbol{\Phi}}$ is a matrix polynomial in λ and from this insight f can eventually be calculated. The solution on the axis is used to choose *the* solution of the Linear Problem from the variety of solutions that emerge from the Bäcklund transform. They differ in the sign of some terms which is due to some square roots. The constructive uniqueness proof of the non-extreme Kerr Black Hole now is complete. One can formulate the theorem:
Provided that the stronger version of the Rigidity theorem holds (see chapter 1), a stationary and axisymmetric vacuum Black Hole with non-degenerate horizon is always a non-extreme Kerr Black Hole.

3.4 Coordinate independent formulation of the Ernst equation and the Linear Problem

3.4.1 The Ernst equation holds everywhere on $\mathcal{M}$

Einstein's equations can be formulated in a coordinate invariant way, which means that the are fullfilled *everywhere* on the spacetime manifold $\mathcal{M}$. (Spacetime singularities do not belong to $\mathcal{M}$).
Now we formulate the Ernst equation as a coordinate invariant equation: Therefore we multiply the Ernst equation in Weyl coordinates from both sides with the conformal factor of $g^{\perp}$, $\frac{1}{\Omega^2} = e^{(2U-2k)}$ of $\perp$, and reformulate:

$$\begin{aligned}
\Re f \frac{1}{\Omega^2}(f,_{\rho\rho} + f,_{\zeta\zeta} + \frac{1}{\rho} f,_{\rho}) &= \frac{1}{\Omega^2}(f,_{\rho}^2 + f,_{\zeta}^2) \\
\Re f \triangle^{(g)} f &= (\frac{1}{\Omega^2}\partial_\rho f \cdot \partial_\rho f + \frac{1}{\Omega^2}\partial_\zeta f \cdot \partial_\zeta f) \\
&= g^{\mu\nu}(\boldsymbol{d}f)_\mu(\boldsymbol{d}f)_\nu \\
\\
\Re f \triangle^{(g)} f &= (\boldsymbol{d}f|\boldsymbol{d}f)^{\perp}. \qquad (3.31)
\end{aligned}$$

Here we have replaced the derivatives in ρ and ζ by the Laplacian of the full metric on the left side and by the square of the gradient of f (with respect to $\perp$) on the right side. The latter formulation is explicitly coordinate independent, as $\Re f$ is a scalar with respect to coordinate transformations as it can be written as $\Re f = e^{2U} = (\xi|\xi)$.
Thus the Ernst equation is valid all over $\mathcal{M}$, it replaces Einstein's equations everywhere. Given an arbitrary $p \in \mathcal{M}$, it follows from the defining properties of a manifold that there is an open neighbourhood of p and local coordinates, such that the Ernst equation written down in this system is valid.

3.4.2 The Linear Problem in invariant formulation

Next to the Inverse Scattering Method in form of the Linear Problem, the calculus of differential forms is a main pillar of the attempt of the proof of the uniqueness of the extrem Kerr Black Hole. By formulating the problem in an invariant way we make sure that it holds all over $\mathcal{M}$.

The Linear Problem in the language of forms Once more we write down the problem in z and $\bar{z}$:

$$\boldsymbol{\Phi},_z = \boldsymbol{U}\boldsymbol{\Phi}, \qquad \boldsymbol{\Phi},_{\bar{z}} = \boldsymbol{V}\boldsymbol{\Phi}.$$

The integrability condition then reads

$$\boldsymbol{U}\boldsymbol{V} + \boldsymbol{U},_{\bar{z}} = \boldsymbol{V}\boldsymbol{U} + \boldsymbol{V}, z$$

or

$$\boldsymbol{U},_{\bar{z}} - \boldsymbol{V},_z + [\boldsymbol{U}, \boldsymbol{V}] = 0.$$

Now a matrix valued differential form $\boldsymbol{\Omega}$ is defined:

$$\boldsymbol{\Omega} := \boldsymbol{U}\boldsymbol{d}z + \boldsymbol{V}\boldsymbol{d}\bar{z}. \tag{3.32}$$

At any time one can substitute z and $\bar{z}$ by ρ and ζ. Due to the dimension of the problem, the manifold where $\boldsymbol{\Omega}$ is defined is $L(2,\mathbb{C})$. We obtain for the exterior derivative

$$\boldsymbol{d\Omega} = \boldsymbol{U}_{,\bar{z}} \cdot \boldsymbol{d}\bar{z} \wedge \boldsymbol{d}z + \boldsymbol{V}_{,z}\, \boldsymbol{d}z \wedge \boldsymbol{d}\bar{z}. \tag{3.33}$$

And for $\boldsymbol{d\Phi}$:

$$\begin{aligned}
\boldsymbol{d\Phi} &= \boldsymbol{\Phi}_{,z}\, \boldsymbol{d}z + \boldsymbol{\Phi}_{,\bar{z}}\, \boldsymbol{d}\bar{z} \\
&= \boldsymbol{U\Phi d}z + \boldsymbol{V\Phi d}\bar{z} \\
&= (\boldsymbol{U d}z + \boldsymbol{V d}\bar{z})\boldsymbol{\Phi} \\
&= \boldsymbol{\Omega\Phi}
\end{aligned}$$

$$\boldsymbol{d\Phi} = \boldsymbol{\Omega\Phi}. \tag{3.34}$$

The latter equation is the desired formulation. In the language of forms, the integrability condition of the problem reads

$$\boldsymbol{d}^2\boldsymbol{\Phi} = 0. \tag{3.35}$$

Written out in components, this equation states that all second partial derivatives commute. We still have to check wether this is equivalent to the integrability condition in matrix form.

$$\begin{aligned}
\boldsymbol{d}^2\boldsymbol{\Phi} &= \boldsymbol{dd\Phi} \\
&= \boldsymbol{d}(\boldsymbol{\Omega\Phi}) \\
&= \boldsymbol{d\Omega} \cdot \boldsymbol{\Phi} - \boldsymbol{\Omega} \wedge \boldsymbol{d\Phi} \\
&= \boldsymbol{d\Omega} \cdot \boldsymbol{\Phi} - \boldsymbol{\Omega} \wedge \boldsymbol{\Omega\Phi} \\
&= (\boldsymbol{d\Omega} - \boldsymbol{\Omega} \wedge \boldsymbol{\Omega})\boldsymbol{\Phi} \\
&= (\boldsymbol{U}_{,\bar{z}} \cdot \boldsymbol{d}\bar{z} \wedge \boldsymbol{d}z + \boldsymbol{V}_{,z}\, \boldsymbol{d}z \wedge \boldsymbol{d}\bar{z} - (\boldsymbol{U d}z + \boldsymbol{V d}\bar{z}) \wedge (\boldsymbol{U d}z + \boldsymbol{V d}\bar{z}))\boldsymbol{\Phi} \\
&= (\boldsymbol{V}_{,z} - \boldsymbol{U}_{,\bar{z}} - \boldsymbol{U} \cdot \boldsymbol{V} + \boldsymbol{V} \cdot \boldsymbol{U}) \cdot \boldsymbol{d}z \wedge \boldsymbol{d}\bar{z} \cdot \boldsymbol{\Phi} \\
&= (\boldsymbol{V}_{,z} - \boldsymbol{U}_{,\bar{z}} + [\boldsymbol{V}, \boldsymbol{U}]) \cdot \boldsymbol{d}z \wedge \boldsymbol{d}\bar{z} \cdot \boldsymbol{\Phi} \\
&= -\Big(\boldsymbol{U}_{,\bar{z}} - \boldsymbol{V}_{,z} + [\boldsymbol{U}, \boldsymbol{V}]\Big) \cdot \boldsymbol{d}z \wedge \boldsymbol{d}\bar{z} \cdot \boldsymbol{\Phi} \\
&= 0.
\end{aligned}$$

All in all the Linear Problem can be formulated in a coordinate independent way. With a matrix valued form $\boldsymbol{\Omega} = \boldsymbol{U d}z + \boldsymbol{V d}\bar{z}$ it reads

$$\boldsymbol{d\Phi} = \boldsymbol{\Omega\Phi} \tag{3.36}$$

$$\boldsymbol{d}^2\boldsymbol{\Phi} = (\boldsymbol{d\Omega} - \boldsymbol{\Omega} \wedge \boldsymbol{\Omega})\boldsymbol{\Phi} = 0. \tag{3.37}$$

The formulation of the Linear Problem in terms of forms states that f can be determined everywhere on $\mathcal{M}$ via the integration of the matrix problem. We can conclude that all defects (discontinuities or worse) result from coordinate defects.

Why Weyl coordinates? So far, all calculations have been carried out in Weyl coordinates. If they are not used there is a further equation next to the Ernst equation for the metric function W [Meinel(1991)]:

$$W, z\bar{z} = 0$$

It can always be solved by setting $W = \rho$ where $\rho := \frac{z+\bar{z}}{2}$. However if one wants to work in a different system and this choice shall be avoided, W enters all equations that lead to the other metric functions. Due to this complications till this day there have been no attemps in the work of the group of Neugebauer and Meinel to use another coordinate system.

Chapter 4

The degenerate horizon and the constructive uniqueness proof of the extreme Kerr Black Hole

We start with the introduction of the concept of the surface gravity, a quantity which is used to characterize the degeneracy of the horizon.
Then we conclude the argumentation given in the course of the previous chapters. Many analytical properties of a degenerate horizon are derived. Finally, we demonstrate how the concepts from above are used to prove two statements: The first one is about the degenerate horizon in Weyl coordinates, and the second one about how the constructive uniqueness proof presented in [Neugebauer and Meinel(2003)] can be extended to the case of a degenerate horizon, which will lead to the extreme Kerr Black Hole.

4.1 The surface gravity

The surface gravity will be a key concept when we want to deal with Black Holes that have a degenerate horizon. Some possibilities of how this quantity can be calculated from a given Black Hole spacetime are provided. Moreover we explain the physical meaning of this term by giving an interpretation. The surface gravity κ of a Black Hole horizon is a scalar with respect to coordinate transformations and it is defined on $\mathcal{H}$ only.

4.1.1 Definition

If one is given two different null vectors a and b with $(a|b) = 0$ then a is proportional to b. This is the case at a Killing horizon. The norm of χ vanishes as well as the norm of the gradient of $N = (\chi|\chi)$. Further on, $(\chi|\boldsymbol{d}N) = \mathcal{L}_\chi N = 0$ (cf. 1.1.2). Thus, both vectors *are proportional on the horizon.* The surface gravity κ is defined as follows:

$$\boldsymbol{d}N = -2\kappa\chi|_{\mathcal{H}}. \tag{4.1}$$

However this formula is useful only in situations where on works with a coordinate system which is valid on the horizon. In Schwarzschild-, Weyl- or Boyer-Lindquist coordinates this equation makes no sense on the horizon.[1] Therefore one would like to solve equation 4.1 for κ. This is done in [Heusler(1996)]. The calculation is carried out in terms of forms:

$$
\begin{aligned}
(\boldsymbol{d}\chi|\boldsymbol{d}\chi) * \chi &= (\boldsymbol{d}\chi|\boldsymbol{d}\chi)\boldsymbol{i}_\chi\eta \\
&= \boldsymbol{i}_\chi((\boldsymbol{d}\chi|\boldsymbol{d}\chi)\eta) \\
&= \boldsymbol{i}_\chi(\boldsymbol{d}\chi \wedge *\boldsymbol{d}\chi) \\
&\overset{\text{L.r.}}{=} \boldsymbol{i}_\chi\boldsymbol{d}\chi \wedge *\boldsymbol{d}\chi + \boldsymbol{d}\chi \wedge \boldsymbol{i}_\chi * \boldsymbol{d}\chi \\
&\overset{\mathcal{L}_\chi\chi=0}{=} -\boldsymbol{d}\boldsymbol{i}_\chi\chi \wedge *\boldsymbol{d}\chi + \boldsymbol{d}\chi \wedge *(\boldsymbol{d}\chi \wedge \chi) \\
&= -\boldsymbol{d}\boldsymbol{i}_\chi\chi \wedge *\boldsymbol{d}\chi - 2\boldsymbol{d}\chi \wedge \omega_\chi \\
&\overset{\mathcal{H}}{=} -\boldsymbol{d}N \wedge *\boldsymbol{d}\chi + 0 \\
&\overset{\mathcal{H}}{=} 2\kappa\chi \wedge *\boldsymbol{d}\chi \\
&\overset{\mathcal{H}}{=} 2\kappa(-) * \boldsymbol{i}_\chi\boldsymbol{d}\chi \\
&\overset{\mathcal{H}}{=} -2\kappa * (-)\boldsymbol{d}N \\
&\overset{\mathcal{H}}{=} -2\kappa * (-)(-)2\kappa\chi \\
&\overset{\mathcal{H}}{=} -4\kappa^2 * \chi.
\end{aligned}
$$

Here, the definition of ω_χ, the Lie property of χ and the identity A-13 have been used. The result is

$$\kappa^2 = -\frac{1}{4}(\boldsymbol{d}\chi|\boldsymbol{d}\chi)|_{\mathcal{H}(K)}. \tag{4.2}$$

An even more useful expression for κ is

$$\kappa^2 = -\frac{1}{4}\triangle N|_{\mathcal{H}(K)}. \tag{4.3}$$

This is deduced by applying the operator $(\chi\boldsymbol{d}^\dagger + \boldsymbol{d} \circ \boldsymbol{i}_\chi)$ to the defining identity for κ:

$$
\begin{aligned}
(\chi\boldsymbol{d}^\dagger + \boldsymbol{d} \circ \boldsymbol{i}_\chi)\boldsymbol{d}N &= -2(\chi\boldsymbol{d}^\dagger + \boldsymbol{d} \circ \boldsymbol{i}_\chi)\kappa\chi \qquad |_{\mathcal{H}} \\
\chi\boldsymbol{d}^\dagger\boldsymbol{d}N + \boldsymbol{d} \circ \boldsymbol{i}_\chi\boldsymbol{d}N &= -2\chi\boldsymbol{d}^\dagger\kappa\chi - 2\boldsymbol{d} \circ \boldsymbol{i}_\chi\kappa\chi \qquad |_{\mathcal{H}} \\
-\chi\triangle N + \boldsymbol{d}(\mathcal{L}_\chi N) &= -2\chi\kappa\boldsymbol{d}^\dagger\chi - 2\chi(\boldsymbol{d}^\dagger\kappa)\chi - 2N\boldsymbol{d}\kappa - 2\kappa\boldsymbol{d}N \qquad |_{\mathcal{H}} \\
-\chi\triangle N + 0 &= 0 - 2(\chi|\chi)(\boldsymbol{d}^\dagger\kappa) - 2N\boldsymbol{d}\kappa - 2\kappa\boldsymbol{d}N \qquad |_{\mathcal{H}} \\
-\chi\triangle N &= -2(\chi|\chi)(\boldsymbol{d}^\dagger\kappa)|_{\mathcal{H}} - 2N\boldsymbol{d}\kappa|_{\mathcal{H}} - 2\kappa\boldsymbol{d}N|_{\mathcal{H}} \\
-\chi\triangle N &= 0 + 0 - 2\kappa(-2\kappa\chi)|_{\mathcal{H}} \\
\triangle N &= -4\kappa^2|_{\mathcal{H}}.
\end{aligned}
$$

[1] In Weyl coordinates one finds $\boldsymbol{d}N = \boldsymbol{d}e^{2\tilde{U}} = \begin{pmatrix} e^{2\tilde{U}}{}_{,\rho} \\ e^{2\tilde{U}}{}_{,\zeta} \\ 0 \\ 0 \end{pmatrix} \overset{?}{=} -2\kappa \begin{pmatrix} 0 \\ 0 \\ \Omega_H \\ 1 \end{pmatrix}$. This makes no sense at all and demonstrates the difficulties one encounters if coordinate defects are present.

The surface gravity of a Kerr Black Hole With formula 4.3 one now can calculate κ for a Kerr Black Hole.

$$\begin{aligned}
\triangle N &= = \frac{1}{\sqrt{|g|}}\partial_\nu(\sqrt{|g|}g^{\nu\mu}\partial_\mu N) \qquad |_{\mathcal{H}}\\
&= \frac{1}{\rho}e^{2(U-k)}\partial_\rho N + e^{2(U-k)}(\partial_{\rho\rho}+\partial_{\zeta\zeta})N \qquad |_{\mathcal{H}}\\
&= (\frac{1}{\rho}e^{2(U-k)}\partial_\rho + e^{2(U-k)}(\partial_{\rho\rho}+\partial_{\zeta\zeta}))(-e^{-2U}\rho^2\omega^2 + e^{2U}(1+a\omega)^2) \qquad |_{\mathcal{H}}\\
&= -\frac{4}{a^2}e^{-2k}, \qquad |_{\mathcal{H}}
\end{aligned}$$

and thus

$$\kappa_{\mathrm{Kerr-b.h.}} = \frac{e^{-k}}{a}. \qquad |_{\mathcal{H}} \tag{4.4}$$

With the formulas for e^{-2k} and a from the appendix one is able to write down the value of κ in terms of M and J.

$$\kappa_{\mathrm{Kerr-b.h.}} = \frac{\sqrt{M^2-\frac{J^2}{M^2}}}{2M\left(M+\sqrt{M^2-\frac{J^2}{M^2}}\right)}. \tag{4.5}$$

In the limit of the extreme Kerr Black Hole, the square root vanishes, and

$$\kappa_{\mathrm{eKbh}} = 0. \tag{4.6}$$

4.1.2 The meaning of the term ‘surface gravity’

The term ‘surface gravity’ is used in astronomy to denote the gravitational acceleration on the surface of a planet or a star in Newton’s theory of gravity. The use in General Relativity can be understood in the following paragraph: It is the force to keep a locally nonrotating observer in place, measured by an observer at infinity. It does not matter whether there is a horizon or not. From this point of view a horizon behaves like a massive body without horizon.

Surface gravity as acceleration to hold a test mass in place, seen from an infinite observer This calculation is similar to [Wald(1984)], chapter 12.5 and chapter 6: It starts with the evaluation of the term $(\chi\wedge \boldsymbol{d}\chi|\chi\wedge\boldsymbol{d}\chi)$:

$$3(\chi\wedge\boldsymbol{d}\chi|\chi\wedge\boldsymbol{d}\chi) = (\chi|\chi)(\boldsymbol{d}\chi|\boldsymbol{d}\chi) + 2\chi_a(\partial_b\chi_c)(\partial^a\chi^b)\chi^c$$

This expression is evaluated on the horizon. Provided that $\kappa\neq 0$, we divide by $(\chi|\chi)$ and evaluate the limit of the resulting equation as we approach the horizon. The left side is then undefined, but de l’Hospital’s rule can be applied and gives the limit zero, as $\chi\wedge\boldsymbol{d}\chi$ is zero over the whole horizon and the gradient of $(\chi|\chi)$ does not become zero due to the assumption $\kappa\neq 0$. $(\boldsymbol{d}\chi|\boldsymbol{d}\chi)$ can be replaced by the definition of κ, formula 4.2:

$$\kappa^2 = \frac{1}{2}\lim_{(\mathcal{H})}\frac{(\chi^a\partial_a\chi^c)(\chi^b\partial_b\chi_c)}{(\chi|\chi)}.$$

The term $\frac{\chi^a \partial_a \chi^c}{(\chi^b \chi_b)}$ can be considered as the acceleration of an orbit of χ (cf. geodesic equation). In the case of a static Black Hole, $\chi = \xi$ and the quantity $\frac{1}{\sqrt{-\xi^a \xi_a}}$ is the redshift factor as seen from infinity. The product κ of acceleration and redshift factor is the force per mass that holds a test mass in place as seen from infinity. Note that the local acceleration becomes infinite on the horizon.
In the case of a rotating Black Hole it is impossible to hold a test mass in place near the horizon, as no stationary particles can exist in the ergoregion. But the term 'surface gravity' is still used.

Important pioneering work in the combination of General Relativity and quantum mechanics has been done by Stephen Hawking. In his picture a Black Hole emits particles with a Black Body spectrum, and κ is a measure for the temperature T of the horizon:

$$T = \frac{\kappa}{2\pi} \quad \Big(= \frac{\hbar\kappa}{2\pi} \Big).$$

A Black Hole with a degenerate horizon thus has zero temperature.
(cf. [Heusler(1996)], chapter 7.)

4.2 The degenerate horizon in Weyl coordinates

We discuss the form of stationary and axisymmetric Killing Horizons in Weyl coordinates, with focus on the degenerate horizon. In the review article [Neugebauer and Meinel(2003)] the Event Horizon of a stationary and axisymmetric Black Hole is an interval on the ζ-axis at $\rho = 0$. In the following section the Linear Problem will be solved for a Black Hole whose horizon is a point on the ζ-axis. To justify this seemingly pathological choice we prove the following statement:
The horizon of a stationary and axisymmetric Black Hole is situated at the point ($\rho = 0$, $\zeta = \zeta_0$) in Weyl-coordinates if and only if the horizon is degenerated.
In Weyl coordinates, a simply connected Killing Horizon appears in two forms:

- A non-degenerate Killing Horizon is an interval on the ζ-axis.
- A degenerate Killing Horizon is an isolated point on the ζ-axis.

4.2.1 Degenerate horizons

The Killing Horizon is defined as the hypersurface on which the norm of a certain Killing vector field χ vanishes. Usually, the gradient of the norm of the Killing vector does not vanish on the horizon, which ensures that this Killing vector will be spacelike at least locally inside the horizon. In this case we speak of a non-degenerate horizon (n-hor). If this is not the case, the horizon is called *degenerate* (d-hor). It could not be found out by the author which one of the below-mentioned features led to the term 'degeneracy'.

Below we give several mathematical conditions for a d-hor.

First, the defining condition: We denote the norm of the Killing vector χ by $(\chi|\chi) = e^{2\tilde{U}}$.

- A killing horizon is degenerate $:\Longleftrightarrow \boldsymbol{d}e^{2\tilde{U}}|_{\mathcal{H}} = 0$.

As $\boldsymbol{d}e^{2\tilde{U}}|_{hor} = \kappa e^{2\tilde{U}}$, this leads to

- The horizon is degenerate if and only if $\kappa = 0$.

Let η denote the Killing vector which generates axisymmetry. For a further condition, we consider the Killing 2-form $\sigma = ((\chi|\chi)(\eta|\eta) - (\chi|\eta)^2)$ (cf. 1.17). This function vanishes on the horizon, as $(\chi|\chi)$ vanishes and as χ on the horizon is orthogonal to η. In Weyl coordinates, we can express σ through metric functions,

$$\begin{aligned}
\sigma &= e^{2\tilde{U}} g_{\phi\phi} - {g_{t\phi}}^2 - 2\Omega_h g_{t\phi} g_{\phi\phi} - {\Omega_h}^2 {g_{\phi\phi}}^2 \\
&= g_{tt} g_{\phi\phi} + 2\Omega_h g_{t\phi} g_{\phi\phi} + {\Omega_h}^2 {g_{\phi\phi}}^2 - {g_{t\phi}}^2 - 2\Omega_h g_{t\phi} g_{\phi\phi} - {\Omega_h}^2 {g_{\phi\phi}}^2 \\
&= (g_{tt} g_{\phi\phi} - {g_{t\phi}}^2) \qquad (4.7)\\
&= det g_{t\phi} \\
&= -\rho^2 \\
&= 0|_{hor}.
\end{aligned}$$

σ turns out to be the square of ρ. Thus on a Killing Horizon one always has $\rho = 0$. On the axis $\eta = 0$ and therefore also $\rho = 0$. But these two regions can be well distinguished by the values of $e^{2\tilde{U}}$ and a.
Now consider the function $\boldsymbol{d}\sigma$:

$$\boldsymbol{d}\sigma = 2(\chi|\eta)\boldsymbol{d}(\chi|\eta) - \boldsymbol{d}e^{2\tilde{U}} \cdot (\eta|\eta) - e^{2\tilde{U}} \cdot \boldsymbol{d}(\eta|\eta).$$

The norm of η is a well behaved function, which is zero only on the axis. If the total mass M of the s&a Black Hole spacetime is greater than 0, the Event Horizon is a non-degenerate hypersurface in $\mathcal{M}$ and the intersection of $\mathcal{H}$ with the axis of symmetry will contain single points only. Everywhere else on $\mathcal{H}$ the function $(\eta|\eta)$ does not equal zero. One can now write down the gradient of σ on the horizon.

$$\Longrightarrow \begin{cases} \boldsymbol{d}\sigma|_{n-hor} &= -(\eta|\eta) \cdot \boldsymbol{d}e^{2\tilde{U}}|_{hor} \\ \boldsymbol{d}\sigma|_{d-hor} &= 0 \end{cases}$$

The latter line is a further characterisation of a degenerate horizon.

One is tempted to write $\boldsymbol{d}\rho^2 = 2\rho\boldsymbol{d}\rho$, but this cannot be true on a n-hor, as in this case the left side is not equal to zero, while the right side would be. We thus can conclude that
The 1-form $\boldsymbol{d}\rho$ exists on the horizon if and only if the horizon is degenerate:

$$\rho \cdot \boldsymbol{d}\rho \to 0 \quad for \quad \rho \longrightarrow 0 \quad \Longleftrightarrow \quad \mathrm{d-hor} \qquad (4.8)$$
$$\rho \cdot \boldsymbol{d}\rho \nrightarrow 0 \quad for \quad \rho \longrightarrow 0 \quad \Longleftrightarrow \quad \mathrm{n-hor} \qquad (4.9)$$

In other words, we have obtaind still another characteristic of a d-hor:
Given a Killing Horizon, then $\boldsymbol{d}\rho$ is a regular 1-form at the horizon if and only if the horizon is degenerate.
Practically speaking, this means that in any suitable coordinate system some partial derivatives of ρ at the horizon will be divergent in the case of a normal and bounded in the case of a degenerate horizon.
Now we derive the form of a degenerate horizon in Weyl coordinates.

4.2.2 The proof

The proof consists of three parts. In the first part we derive a condition equivalent to "ζ is constant on the horizon". In the second part we write down some relevant identities for a degenerate horizon. The third part utilizes the first two to arrive at a proof.

We start with assumptions on the horizon: We assume the part of the horizon that lies in $\perp$ to be a compact subset of $\perp$ and that any one-dimensional volume of the union of the horizon with the axis of symmetry will give zero. ("The horizon shields a part of the axis."). Locally this intersection $\mathcal{H} \cap \perp$ is an intervall I on the ζ-axis. Now we introduce local coordinates h, t, ϕ on $I \times \mathbb{R} \times [0, 2\pi]$. These coordinates shall form a basis:

$$\begin{aligned} (\boldsymbol{d}h|\boldsymbol{d}h)^{\perp} \neq 0; \quad (\boldsymbol{d}t|\boldsymbol{d}t)^{\perp} \neq 0; \quad (\boldsymbol{d}\phi|\boldsymbol{d}\phi)^{\perp} \neq 0; \\ (\boldsymbol{d}h|\boldsymbol{d}t)^{\perp} = 0; \quad (\boldsymbol{d}h|\boldsymbol{d}\phi)^{\perp} = 0; \quad (\boldsymbol{d}t|\boldsymbol{d}\phi)^{\perp} = 0. \end{aligned} \tag{4.10}$$

h is an inextendible coordinate on $\mathcal{H}$, but can be also considered as a function on $\perp$. If one now considers the line

$$\zeta_0(h) := \zeta(h)|_{\mathcal{H}}$$

one sees that on the horizon

$$\boldsymbol{d}\zeta = \zeta_{,h}\, \boldsymbol{d}h|_{\mathcal{H}}. \tag{4.11}$$

Whether ζ is constant on $\mathcal{H}$ or not can be seen by whether $\zeta_{,h}$ is zero or not. This in turn can be expressed in the following way:

$$\begin{aligned} (\boldsymbol{d}\rho|\boldsymbol{d}\rho)^{\perp} &= (\boldsymbol{d}\zeta|\boldsymbol{d}\zeta)^{\perp} \overset{\mathcal{H}}{=} (\zeta_{,h}\, \boldsymbol{d}h|\zeta_{,h}\, \boldsymbol{d}h)^{\perp} \\ &= (\zeta_{,h})^2 (\boldsymbol{d}h|\boldsymbol{d}h)^{\perp}. \end{aligned} \tag{4.12}$$

Thus we can express $\zeta_{,h}$ through the gradient of ρ, and because $\perp$ is Riemannian:

$$\zeta_{,h} = 0 \iff (\boldsymbol{d}\rho|\boldsymbol{d}\rho)^{\perp}|_{\mathcal{H}} = 0 \iff \boldsymbol{d}\rho|_{\mathcal{H}} = 0. \tag{4.13}$$

So far we only know something about the gradient of $\boldsymbol{d}\rho^2$, namely

$$\boldsymbol{d}\rho^2|_{\mathcal{H}} = 0 \iff \mathrm{d}-\mathrm{hor.}$$

and the formula

$$\boldsymbol{d}\rho^2 = 2\rho\boldsymbol{d}\rho|_{\mathcal{H}}$$

is valid if and only if the horizon is degenerate. The key to the proof will be the Einstein equation for ρ:

$$\triangle^{\perp}\rho = 0.$$

Note that the "odd Laplacian" satisfies the Leibnitz rule. Therefore we also have

$$\triangle^{\perp}\rho^2 = 0.$$

(The opposite would in general not be true: if a real function f is C^2, its square root (if it exists at all) may not be. Just take $f(x) = x$.)

Now we conclude the proof:

"$\Longleftarrow$": If ζ is constant on the Horizon, $\zeta_{,h} = 0$ and thus $\boldsymbol{d}\rho|_{\mathcal{H}} = 0$ and the horizon is degenerate.

"$\Longrightarrow$": We start with the Laplace equation for ρ and ρ^2, respectively, and we express $\triangle^{\perp}$ as $\triangle^{\perp} = *\boldsymbol{d} * \boldsymbol{d}$. (We should write $\boldsymbol{d}^{\perp}$ instead of $\boldsymbol{d}$, but this is omitted up to the final result.)

$$\begin{aligned} \triangle^{\perp}\rho = *\boldsymbol{d} * \boldsymbol{d}\rho &= 0; \quad \triangle^{\perp}\rho^2 = *\boldsymbol{d} * \boldsymbol{d}\rho^2 = 0 \\ \Longleftrightarrow \boldsymbol{d} * \boldsymbol{d}\rho &= 0 = \quad \boldsymbol{d} * \boldsymbol{d}\rho^2. \end{aligned} \tag{4.14}$$

Now we consider a degenerate horizon. Then, and only then we can make use of the equation $\boldsymbol{d}\rho^2 = 2\rho\boldsymbol{d}\rho|_{\mathcal{H}}$, for substituting $\boldsymbol{d}\rho^2$:

$$\begin{aligned} 0 &= \boldsymbol{d} * \boldsymbol{d}\rho^2 \\ &= \boldsymbol{d} * 2\rho\boldsymbol{d}\rho \\ &= 2\boldsymbol{d}(\rho * \boldsymbol{d}\rho) \\ &\overset{L.r.}{=} 2\boldsymbol{d}\rho \wedge *\boldsymbol{d}\rho + 2\rho\boldsymbol{d} * \boldsymbol{d}\rho. \end{aligned}$$

The second term vanishes on $\mathcal{H}$, and we can write down the final condition

$$\boldsymbol{d}\rho \wedge *\boldsymbol{d}\rho = 0|_{d-hor} \tag{4.15}$$

As we calculated $\triangle^{\perp}\rho$, we have to consider a two-dimensional Riemannian space rather than a (3+1)-Minkowski space. In general, the Laplacian for an arbitrary $p-$form is defined as

$$\triangle := -[\boldsymbol{d}^{\dagger} \circ \boldsymbol{d} + \boldsymbol{d} \circ \boldsymbol{d}^{\dagger}].$$

The correct final condition reads

$$\boldsymbol{d}^{\perp}\rho \wedge *\boldsymbol{d}^{\perp}\rho = 0|_{d-hor}. \tag{4.16}$$

This term is a 2-form and it depends only on one scalar function, which turns out to be $(\boldsymbol{d}\rho|\boldsymbol{d}\rho)^{\perp}$:

$$\begin{aligned} \boldsymbol{d}^{\perp}\rho \wedge *\boldsymbol{d}^{\perp}\rho &= 0 \\ &= (\boldsymbol{d}^{\perp}\rho)_a \boldsymbol{d}x^a \wedge \sqrt{|\tau|}\epsilon_{fb}(\boldsymbol{d}^{\perp}\rho)^f \cdot \boldsymbol{d}x^b \\ &= (\boldsymbol{d}^{\perp}\rho)_a(\boldsymbol{d}^{\perp}\rho)^f \cdot \sqrt{|\tau|}\epsilon_{fb} \cdot \boldsymbol{d}x^a \wedge \boldsymbol{d}x^b \\ &= 2\sqrt{|\tau|} \cdot (\boldsymbol{d}^{\perp}\rho)_a(\boldsymbol{d}^{\perp}\rho)^a \cdot \boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2 \\ &= 2\sqrt{|\tau|} \cdot (\boldsymbol{d}\rho)|\boldsymbol{d}\rho)^{\perp} \cdot \boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2. \end{aligned} \tag{4.17}$$

Now we know that only in the degenerate case we do have

$$(\boldsymbol{d}\rho|\boldsymbol{d}\rho)^{\perp}|_{\mathrm{d-hor}} = 0,$$

and thus (by equation 4.12)

$$\boldsymbol{d}\zeta|_{\mathrm{d-hor}} = 0,$$

which means that ζ is constant on the horizon.

4.3 The constructive uniqueness proof

Continuity in Weyl coordinates The Linear Problem in terms of forms reads

$$\boldsymbol{d\Phi} = \boldsymbol{\Omega\Phi}$$

(see chapter 3). This formulation is coordinate independent, and for each $p \in M$ there is at least one chart (coordinate system) for which both equations hold in a neighbourhood of p when written down in this system. Then, $\boldsymbol{\Phi}$ is a C^2-function, and the Ernst equation will also hold in a certain neighbourhood, as it is the integrability condition of the Linear Problem. Thus in this system we have a unique C^2-Ernst potential and a C^2-$\boldsymbol{\Phi}$.

First we introduce the half-side axis Ernst potential which will be used to define f on a degenerate horizon in Weyl coordinates.
The Ernst potential is a scalar and thus remains unchanged under a regular coordinate transform. But its derivatives will change. Now let a s&a vacuum spacetime with a degenerate horizon at $(0,0)$ be given. On the axis, f is continous and we split the axis potential into

$$f(\zeta) = f_+ + f_-, \quad \text{where} \quad f_+ = \begin{cases} f, & \zeta \in \mathcal{A}^+ \\ 0, & \zeta \in \mathcal{A}^- \end{cases}, \quad \text{and} \quad f_- = \begin{cases} 0, & \zeta \in \mathcal{A}^+ \\ f, & \zeta \in \mathcal{A}^- \end{cases}, \tag{4.18}$$

with both f_+ and f_- half-side-continous in $(0,0)$. In the degenerate case f is not defined on the horizon in Weyl coordinates, and continuity cannot be established by *defining* f in $(0,0)$, because this leads to Minkowski space. We conclude that f in Weyl coordinates is not continous on $\mathcal{H}$ and thus the limits of f_+ and f_- at $(0,0)$ must be different from another and we denote these two limits by f_1 (limit of f_+) and f_2 (limit of f_-). Furthermore they are both imaginary, because e^{2U} has to vanish, if one approaches the horizon on the axis.

Second let (μ,ν) be local coordinates at one junction of $\mathcal{H}$ with the axis in which the Linear Problem holds. Then we consider $\boldsymbol{\Phi}(\rho(\mu,\nu),\zeta(\mu,\nu))$. The junction of $\mathcal{H}$ with the axis can be defined invariantly as '$(\eta = 0) \cap (e^{2\tilde{U}} = 0) \cap (\chi \neq 0)$'.
The path $\mathcal{A}^+\mathcal{B}\mathcal{A}^-$ in $\mathcal{M}$ has a coordinate independent meaning as well. On this path $\mathcal{A}^+\mathcal{B}\mathcal{A}^-$, $\boldsymbol{\Phi}$ is C^2 in local coordinates . The transformation to Weyl coordinates contracts the degenerate horizon into a single point, but during the subsequent calculation we refer to the underlying *invariant* parametrisation of the horizon. As in the case of f, the quantity $\boldsymbol{\Phi}(\rho = 0, \zeta = 0)$ is not defined on a d-hor, but we can give meaning to the directional limits $\boldsymbol{\Phi}_{(\text{direction})}(\rho = 0, \zeta = 0)$ (coming from the horizon or from the axis) via the underlying C^2-structure and because of the invariance of $\mathcal{A}^+\mathcal{B}\mathcal{A}^-$. These directional limits are sufficient for the subsequent calculation, which is performed only on $\mathcal{A}^+\mathcal{B}\mathcal{A}^-$.

The Linear Problem and the degenerate horizon Now, let us assume that $\boldsymbol{\Phi}$ and $\widetilde{\boldsymbol{\Phi}}$ are continous from one side only. Via the underlying structure, we have the same matrix conditions as in the nondegenerate case. But now $K_1 = 0 = K_2$

hence $\alpha_1 = \alpha_2 = 2i\Omega_h K =: \alpha(K)$. The formulas for F and G become (cf. 3.28):

$$F = \frac{(\alpha + f_2)^2 - 1}{(\alpha + f_1)(\alpha + f_2) - 1}; \quad G = \frac{f_2 - f_1}{(\alpha + f_1)(\alpha + f_2) - 1}. \tag{4.19}$$

The modified conditions 3.27 apply again, which leads to several constraints since the equation must hold for every power of α:

$$(\alpha - f_1)(\alpha + f_1)(\alpha + f_2) - (\alpha - f_1) = f_2^2(\alpha + f_2) - f_2^2(\alpha + f_1)(\alpha - f_2)(\alpha + f_2)^2 - (\alpha - f_2), \tag{4.20}$$

for all α. From that we have: $\alpha^1 : f_1^2 = f_2^2$ and $\alpha^0 : (f_1 - f_2) = -f_1^2(f_1 - f_2)$. These constraints for the $f_i s$ have three possible solutions:
a) $f_1 = 0 = f_2$,
b) $f_1 = f_2 \neq 0$,
c) $f_1 = -f_2 \neq 0$.
Now one considers the Ernst potential for positive ζ.

$$A^+) \quad f = \frac{G+1}{F} = \frac{f_2 - f_1 - 1 + (\alpha + f_1)(\alpha + f_2)}{(\alpha + f_2)^2 - 1}. \tag{4.21}$$

If one chooses the $f_i s$ as in a) or b) the Ernst potential f turns out to be identically one, which describes the Minkowski space. But this situation must be excluded. Thus constraint c) must hold. From the α^0-equation we get $f_1^2 = -1$ and hence

$$f_1 = \pm i, f_2 = \mp i. \tag{4.22}$$

If one substitutes these values into f, one obtains

$$f_{(1)/(2)}(K) = \frac{\alpha + 1 \pm i}{\alpha + 1 \mp i}. \tag{4.23}$$

The solutions are reciprocal to each other. It is a general result that, if f is a given solution of the Ernst equation, $1/f$ is also a solution, provided it is defined. This can be seen by a straightforward calculation (apply the Ernst equation to $1/f$ and rearrange the terms).
Via analytic continuation we replace K by ζ and obtain the Ernst potential for the upper part of the axis:

$$f_{(1)/(2)}(\zeta) = \frac{\zeta - \frac{i}{2\Omega} \pm \frac{1}{2\Omega}}{\zeta - \frac{i}{2\Omega} \mp \frac{1}{2\Omega}}. \tag{4.24}$$

Note that Ω is the only remaining parameter of the two solutions. We have found f in terms of the angular velocity of the horizon of the Black Hole. An asymptotic expansion of f leads - as is has been carried out in the nondegenerate case - again to an algeabric connection between the total mass M and Ω. One finds

$$M = \pm \frac{1}{2\Omega}. \tag{4.25}$$

Since M must be positive, not all ranges of Ω lead to a physically reasonable solution. To get a positive M, we have to choose a negative Ω for $f_{(1)}$ and a positive Ω for $f_{(2)}$.

	$\Omega > 0$	$\Omega < 0$
$f_{(1)}$	$M < 0$ not allowed	$M > 0$
$f_{(2)}$	$M > 0$	$M < 0$ not allowed

Figure 4.1: Physically reasonable ranges of Ω.

Finally, we have for positive ζ:

$$f_{(1)/(2)}(\zeta) = \begin{cases} \frac{\zeta - M + iM}{\zeta + M + iM}, & \Omega \leq 0 \\ \\ \frac{\zeta - M - iM}{\zeta + M - iM}, & \Omega \geq 0 \end{cases} \qquad \zeta > 0. \tag{4.26}$$

The Ernst potential for negative ζ is analogously determined by making use of the formulas for $\mathcal{A}^-$. These two solutions are the only possible axis values of the Ernst potential which lead to a stationary, axisymmetric Black Hole with positive total mass. Of course these solutions are known as the extreme Kerr Black Holes with the angular momentum $J = \pm M^2$. The knowledge of the axis potential uniquely determines the Ernst potential everywhere (cf. chapter 2). The global solution can be calculated by a Bäcklund transformation, in the same way as it has been done in the non-degenerate case.

4.4 Towards a uniqueness theorem

The result of the previous section can be summarized as follows:
If a single point on the ζ-axis is set as Killing Horizon in an s&a vacuum spacetime, then there is a unique physically reasonable solution, the extreme Kerr Black Hole.
Starting from boundary conditions we could derive a Black Hole spacetime which respects them. To obtain this result, the existing proof of Neugebauer and Meinel [Neugebauer and Meinel(2003)] had to be modified.
One has to ask the question whether the assumption of a single-point-horizon in Weyl coordinates is reasonable. Of course the choice is affirmed by the result, but it is desirable to know it a priori. This could be achieved by proving, that *a Killing Horizon in a s&a spacetime is degenerate if and only if it is a point on the ζ-axis in Weyl coordinates.*
The final step is then to prove, that
an Event Horizon of a stationary spacetime is a Killing Horizon.
This means that the case of a degenerate Horizon has to be included in the rigidity theorem.

To establish a uniqueness theorem for the extreme Kerr Black Hole, the order of these three statements has to be reversed (cf. conclusion).

Conclusion

The quest for spacetimes which possess symmetries led to the concept of a Killing vector field. It also turned out to be a crucial concept for the description of a stationary Black Hole horizon. For stationary vacuum spacetimes the existing uniqueness theorems classify almost all known Black Hole solutions.
To take a step towards the completion of these theorems we introduced a special coordinate system, the Weyl coordinates, and Einstein's equations in this system, the Ernst equation. With them and with the Linear Problem we explained many details of the construction of a stationary and axisymmetric solution to Einstein's equation, the non-degenerate Kerr Black Hole.
This constructive uniqueness proof could be extended to a stationary and axisymmetric vacuum Black Hole with a degenerate horizon:
The extreme Kerr Black Hole is the only stationary and axisymmetric vacuum Black Hole with a simply connected degenerate horizon.

However more work has to be done in order to introduce this proof into the family of uniqueness theorems for stationary Black Holes.
A first result on this way is that a single degenerate Killing horizon is always represented by a point on the ζ-axis in Weyl coordinates. This could be proved in this thesis. Further on, an extended version of the Rigidity theorem – including degenerate horizons – has to be proved. Then the statements of the uniqueness proof read as follows (For the italic statements a proof has been given in this thesis.):

- The Event Horizon of a stationary, degenerate vacuum Black Hole is a Killing Horizon. The spacetime is either static or axisymmetric (Rigidity theorem).
- *In the axisymmetric case a simply connected degenerate Killing Horizon is a single, isolated point on the ζ-axis in Weyl coordinates.*
- *From the solution of a Linear Problem and from a Bäcklund transformation of the Minkowski space it follows that the extreme Kerr Black Hole is the only solution of the Ernst equation in the case of a single point Killing Horizon.*

We should note that the chain of arguments that leads to the extension of the uniqueness theorems is expected to be true, but that is has happened often that both nature and mathematics turn out to be more complicated than one anticipates.

Appendix A

Differential forms - mathematical and intuitive aspects

The mathematical origin of differential forms is briefly introduced, the description is intended to be more instructive than rigorous. Some examples of application and suggestions for imagination follow. Many important initial contributions to this topic were made in the beginning of the last century by Elie Cartan (1869-1951), a french mathematician. Sometimes this topic is called “exterior calculus” which refers to the fact that vectors, tensors and other objects lie in the tangent bundle rather than on the manifold itsself.

In conjunction with usual techniques and notations of differential geometry the calculus of differential forms provides a powerful tool in both, calculation and imagination.

Basic notations and definitions

Vectors and covectors Vector fields on a manifold $\mathcal{M}$ can be considered as directional derivatives that lie in the tangent bundle $T\mathcal{M}$. At any given $q \in \mathcal{M}$ they map covectors into numbers. All vectors at q live in the tangent space at q, $T_q\mathcal{M}$, which in this report is a vector space with Euklidean metric (for Riemann manifolds) or with Lorentzian metric (for Einstein manifolds), respectively. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, ...\boldsymbol{e}_n\}$ be a vector basis in $T_q\mathcal{M}$. Then one can introduce the *dual basis* $\{\boldsymbol{dx}^1, \boldsymbol{dx}^2, ...\boldsymbol{dx}^n\}$ of covectors in $T_q\mathcal{M}$. The basic relation between vector and covector basis is their scalar product $(.|.)$ (interior product) on $\mathcal{M}$,

$$(\boldsymbol{dx}^i|\boldsymbol{e}_j) = \delta^i_j. \tag{A-1}$$

Vectors are usually illustrated as arrows pointing in a certain direction, and covectors can be considered as a stack of hypersurfaces with orientation. The interior product between both, vector and covector, can be considered as the number of slices pieced by the vector. But of course this must not be taken literally as neither the length (not the norm!) of a vector nor the number of slices of a covector have a physical and coordinate independent meaning. The metric has not played a role so far. It is not needed for the calculation of the interior product of a vector with a covector. But of course the metric can be used to calculate the correspondig covector to a given vector and vice versa. (See next paragraph.) Arbitrary vectors and covectors are obtained by linear combinations of the basic (co)vectors. Co-

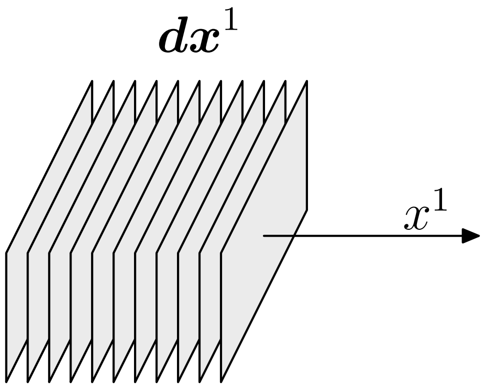

Figure 2: How one can imagine a basic co-vector (1-form).

vectors are typical examples of differential forms. All 1-forms *are* covectors. In the following paragraph we will see how more general forms are constructed.

Differential forms in general

The wedge product Sometimes it is very helpfull to think of a vector as a map which maps covectors into $\mathbb{R}$ and to think of a covector as a map which maps vectors into $\mathbb{R}$. As there is a tensor product for vectors, there is also a tensor product for covectors, here is an example of the basis of (co)vectors:

$$(\boldsymbol{dx}^i \otimes \boldsymbol{dx}^j \otimes \boldsymbol{dx}^k | \boldsymbol{e}_l \otimes \boldsymbol{e}_m \otimes \boldsymbol{e}_n) = (\boldsymbol{dx}^i|\boldsymbol{e}_l)(\boldsymbol{dx}^j|\boldsymbol{e}_m)(\boldsymbol{dx}^k|\boldsymbol{e}_n) = \delta^i_l \delta^j_m \delta^k_n.$$

The basic covectors live in the dual of the tangent bundle, $T^*\mathcal{M}$. To construct an effective tool for manipulations of tensor equations, the *antisymmetric tensor product* or *wedge product* "$\wedge$" for basic covectors is introduced:

$$\boldsymbol{dx}^i \wedge \boldsymbol{dx}^j := \frac{1}{2}\Big(\boldsymbol{dx}^i \otimes \boldsymbol{dx}^j - \boldsymbol{dx}^j \otimes \boldsymbol{dx}^i\Big)\ ^{(2)}. \tag{A-2}$$

An immediate consequence is

$$\boldsymbol{dx}^i \wedge \boldsymbol{dx}^i \equiv 0. \tag{A-3}$$

For the wedge product of p factors one considers all permutations of tensor products and sums them up by respecting the order of the permutation, in the same way as it is done for a matrix determinant. Now we have constructed the *basic p-forms.* The linear hull of them constitutes the space of p-forms. Due to the antisymmetry, the highest possible degree of a form is n, the dimension of the Manifold. Forms of higher degree are set zero by definition. A 0-form is a scalar with respect to coordinate transformations by definition. The *degree p* of a form is the number of basic forms in the wedge product.

[2] The prefactor may differ from author to author.

The components of p-forms Now that we have introduced the basic forms, let us draw our attention to their components. First an example: From the three covectors $\boldsymbol{a} = a_\mu \boldsymbol{dx}^\mu$, $\boldsymbol{c} = b_\mu \boldsymbol{dx}^\mu$ and $\boldsymbol{c} = c_\mu \boldsymbol{dx}^\mu$ we construct the 3-form

$$\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} = a_\mu b_\nu c_\rho \; \boldsymbol{dx}^\mu \wedge \boldsymbol{dx}^\nu \wedge \boldsymbol{dx}^\rho.$$

Any permutation of μ, ν and ρ gives - up to sign - the same term, and in cases where the three indices are not totally different from another, this term does not contribute to the form. Thus one can write

$$a \wedge b \wedge c = a_{[\mu} b_\nu c_{\rho]} \; \boldsymbol{dx}^\mu \wedge \boldsymbol{dx}^\nu \wedge \boldsymbol{dx}^\rho \quad .$$

One can also say that a $p-$form is a *completely antisymmetric covariant tensor field of rank p.* But this is only one side of the medal, on the other side there are the basic forms.

Examples of forms of higher degree

- The electromagnetic field tensor $\boldsymbol{F}$ is antisymmetric and can thus be considered as a 2-form.
- The 3-form of charge density $*\boldsymbol{j}$ in the Maxwell theory: It is an expression of electric charge and currents in spacetime in terms of forms.
- The totally antisymmetric $\boldsymbol{\epsilon}$-symbol is a simple example of a n-form, another one is the volume-n-form

$$\eta_{\mu_1 \mu_2 \dots \mu_n} = \sqrt{|g|} \epsilon_{\mu_1 \mu_2 \dots \mu_n}.$$

Troughout this text, forms are written in bold symbols, however the components are written normally. An exeption is made for 1-forms, as on manifolds with metric *any* 1-form can be written as a vector and vice versa.

The calculus of p-forms

Here we want to demonstrate the differential calculus of forms and other manipulations.

The interior multiplication Consider a p-form as a map of p vectors into $\mathbb{R}$ (or $\mathbb{C}$). If one instead maps only one vector, the remaining object will be a $p-1$-form. We define the *interior multiplication* $\boldsymbol{i}_X$ of a p-form $\boldsymbol{\alpha}$ with a vector X as

$$(\boldsymbol{i}_X \boldsymbol{\alpha})(X_1, X_2, ..., X_n) := \boldsymbol{\alpha}(X, X_1, X_2, ..., X_n). \tag{A-4}$$

On manifolds with metric one can express the norm of a vector X as an interior multiplication:

$$g(X, X) = X^\mu X_\mu = (X|X) = \boldsymbol{i}_X X. \tag{A-5}$$

The interior multiplication of a vector with a scalar (0-form) is zero by definition.

The exterior derivative The partial derivative of a form neither respects the degree of a form nor its antisymmetry nor the independence of coordinates. However there is a proper derivative operator for forms with remarkable properties, the *exterior derivative.* It increases the degree of the form by 1. The exterior derivative of a scalar ϕ is given by

$$\boldsymbol{d}\phi := \partial_\mu \phi \, \boldsymbol{dx}^\mu. \tag{A-6}$$

and by the help of this definition one can denote the exterior derivative for a p-form $\boldsymbol{\alpha} = \alpha_{\nu_1\nu_2\ldots\nu_p}\boldsymbol{dx}^{\nu_1} \wedge \boldsymbol{dx}^{\nu_2} \wedge \ldots \wedge \boldsymbol{dx}^{\nu_p}$:

$$\boldsymbol{d\alpha} := \partial_\mu \alpha_{\nu_1\nu_2\ldots\nu_p} \, \boldsymbol{dx}^\mu \wedge \boldsymbol{dx}^{\nu_1} \wedge \boldsymbol{dx}^{\nu_2} \wedge \ldots \wedge \boldsymbol{dx}^{\nu_p}. \tag{A-7}$$

Hence the exterior derivative of a n-form is zero.
On manifolds without torsion one can show ([Wald(1984)], S. 429), that
any derivative operator on $\mathcal{M}$ can be used to calculate the (unique) exterior derivative.
This is due to the fact that vanishing torsion is equivalent to a symmetric affine connection, and symmetric objects summed up with antisymmetric ones give zero. Hence it does not matter which derivative operator is used.
For the same reason the exterior derivative, when applied twice, gives zero (Consider for example a C^2-1-form $\boldsymbol{\alpha}$):

$$\begin{aligned} \boldsymbol{d} \circ \boldsymbol{d\alpha} &= \boldsymbol{d}\Big(\partial_\mu \alpha_\nu \, \boldsymbol{dx}^\mu \wedge \boldsymbol{dx}^\nu\Big) \\ &= \partial_\rho \partial_\mu \alpha_\nu \, \boldsymbol{dx}^\rho \wedge \boldsymbol{dx}^\mu \wedge \boldsymbol{dx}^\nu \\ &= \partial_\mu \partial_\rho \alpha_\nu \, \boldsymbol{dx}^\rho \wedge \boldsymbol{dx}^\mu \wedge \boldsymbol{dx}^\nu \\ &= 0. \end{aligned} \tag{A-8}$$

This provides an elegant way to express the integrability condition of a Linear Problem in a coordinate independent way. (See chapter 3.4)

The dual form and the co-derivative Up to this point all operations can be performed even on manifolds without metric. The *dual* of a form however requires a metric tensor on $\mathcal{M}$. Let s denote the number of negative eigenvalues of the metric. To every p-form $\boldsymbol{\alpha}$ one can assign its *Hodge dual*, a $(n-p)$-form $*\boldsymbol{\alpha}$ which reads in components

$$(*\boldsymbol{\alpha})_{\mu_{p+1}\ldots\mu_n} = \frac{1}{p!}\eta\mu_1\ldots\mu_n\alpha^{\mu_1\ldots\mu_p}. \tag{A-9}$$

Important relations that include the Hodge dual are

$$*1 = \eta; \qquad *K = \boldsymbol{i}_K * 1 = \boldsymbol{i}_K\eta,$$

where K is a vector or its assigned 1-form. Consider for example Maxwell's equations which contain the Faraday tensor as well as its dual:

$$\begin{aligned} \boldsymbol{dF} &= 0, \\ \boldsymbol{d} * \boldsymbol{F} &= 4\pi * \boldsymbol{j}. \end{aligned}$$

An interpretation will be given below.

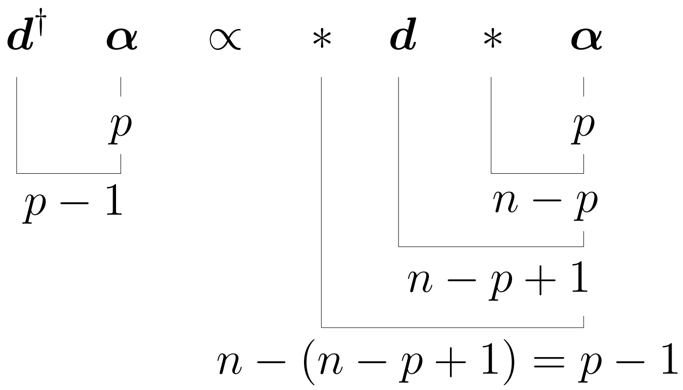

Figure 3: The degrees during the calculation of $\boldsymbol{d}^\dagger\boldsymbol{\alpha}$.

With the help of the *co-derivative* $\boldsymbol{d}^\dagger$ one can reduce the degree of a form by 1. It is defined by taking in turn the exterior derivative $\boldsymbol{d}$ and twice the dual $*$, and for a $p-$form $\boldsymbol{\alpha}$ it is

$$\boldsymbol{d}^\dagger\boldsymbol{\alpha} := -(-1)^{n(p+1)+s} * \boldsymbol{d} * \boldsymbol{\alpha}. \tag{A-10}$$

Here

- n denotes the dimension of the manifold,
- p is the degree of the form $\boldsymbol{\alpha}$ and
- s is the number of negative eigenvalues of the metric.

Let $\boldsymbol{\alpha}$ be a $p-1$-form and $\boldsymbol{\beta}$ be a p-form. Then $\boldsymbol{d}^\dagger$ is the adjoint of $\boldsymbol{d}$ with respect to the inner product

$$\langle\, .\, ,\, .\, \rangle = \int_{\mathcal{M}} (\, .\, |\, .\,)\eta : \qquad \langle \boldsymbol{d\alpha}, \boldsymbol{\beta}\rangle = \langle \boldsymbol{\alpha}, \boldsymbol{d}^\dagger\boldsymbol{\beta}\rangle. \tag{A-11}$$

We will make use of two cases:

- a) The (3+1)-Minkowski space, $n = 4$ and $s = 1$, hence
  $\boldsymbol{d}^\dagger\boldsymbol{\alpha} := -(-1)^{4(p+1)+1} * \boldsymbol{d} * \boldsymbol{\alpha} = *\boldsymbol{d} * \boldsymbol{\alpha}$.
- b) The 2-dimensional Riemann space, where $n = 2$ and $s = 0$, and hence
  $\boldsymbol{d}^\dagger\boldsymbol{\alpha} := -(-1)^{2(p+1)+0} * \boldsymbol{d} * \boldsymbol{\alpha} = - * \boldsymbol{d} * \boldsymbol{\alpha}$.

An application of this definition is the expression for the Laplacian of a scalar ϕ in coordinates:

$$\begin{aligned}
\triangle\phi &:= -[\boldsymbol{d}^\dagger \circ \boldsymbol{d} + \boldsymbol{d} \circ \boldsymbol{d}^\dagger]\phi \\
&\overset{\boldsymbol{d}^\dagger\phi=0}{=} \boldsymbol{d}^\dagger \circ \boldsymbol{d}\phi \\
&= (-)(-)(-)^{n+s} * \boldsymbol{d} * \boldsymbol{d}\phi \\
&= (-)^{n+s} * \boldsymbol{d} * \boldsymbol{d}\phi \\
&= \frac{1}{\sqrt{|g|}}\partial_\mu\Big(\sqrt{|g|}\partial^\mu\phi\Big).
\end{aligned} \tag{A-12}$$

Finally we denote some identities for a vector X and a p-form $\boldsymbol{\alpha}$ that contain the operations that have been discussed so far.

$$\boldsymbol{i}_X\boldsymbol{\alpha} = - * (X \wedge *\boldsymbol{\alpha}); \qquad \boldsymbol{i}_X * \boldsymbol{\alpha} = *(\boldsymbol{\alpha} \wedge X). \tag{A-13}$$

Integration of a form Let $\mathcal{M}$ be an orientable n-dimensional manifold and let U be a part of $\mathcal{M}$ where there is a map $\psi : U \longrightarrow \mathbb{R}^n$. Then the integral of a n-form $\boldsymbol{\alpha} = \alpha \cdot \boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2 \wedge ... \wedge \boldsymbol{d}x^n$ on U is defined as

$$\int_U \boldsymbol{\alpha} := \int_{\psi[U]} \alpha dx^1 dx^2 ... dx^n. \tag{A-14}$$

The right side is the Riemann or Lebesgue integral of the function α on the image $\psi[U]$ which is a subset of $\mathbb{R}^n$. This definition is invariant of the choice of the coordinate system. If the degree is smaller than n, the dimension of the volume also has to be reduced. A covector can be integrated along a curve, a 2-form over a 2-surface and so on. The result can be interpreted as length, as flow, as charge and so on.

The intuitive aspect of differential forms

Most of the concepts presented here can be found in
[Charles W. Misner and Wheeler(2000)].

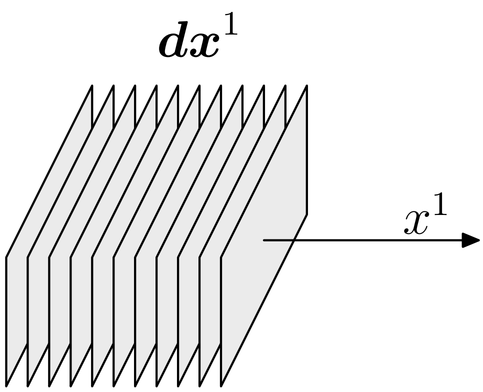

Figure 4: How one can imagine the gradient of the coordinate x^1.

1-forms A 1-form $\boldsymbol{d}\phi$ represents both, the unspecified directional derivative and a precise mathematical notion of the ‘differential’ of a function ϕ, as it is often used by physicists.
Given a scalar field f and vector field v, then the directional derivative of f with respect to v is given by

$$\partial_v f := (\partial_\mu f) v^\mu \equiv (\boldsymbol{d}f | v).$$

The exterior derivative of a scalar is its unspecified directional derivative, also called the gradient, because its *components* are $\partial_\mu f$. On the other side one can consider a vector as directional derivative for a special direction.
The basic 1-forms $\{\boldsymbol{dx}^i\}$ of the coordinates are their directional derivatives, and any 1-form can be written as linear combination of them. A common picture of the gradient is a stack of orientated hypersurfaces.
([Charles W. Misner and Wheeler(2000)], or figure 4.)
Now consider a particle on the x-axis on which a force $F(x)$ acts. Then the energy E that the particle gains from the force field when traveling from a to b is

$E = \int_a^b F(x)dx$. Now one can use the calculus of forms and define a work form $\boldsymbol{w} := F(x) \cdot \boldsymbol{dx}$ and hence

$$E = \int_a^b F(x)dx = \int_{[a,b]} \boldsymbol{w}.$$

Now compare the two expressions

$$dE = F(x)dx; \qquad \boldsymbol{w} := F(x) \cdot \boldsymbol{dx}.$$

The first one is not defined while the second one is. Both expressions can be interpreted in the same way. The corresponding integrals are both defined.
This is a way to handle 'differentials'. The basic 1-forms are in this sense the sucessors of the differentials dx^1 ... dx^n.

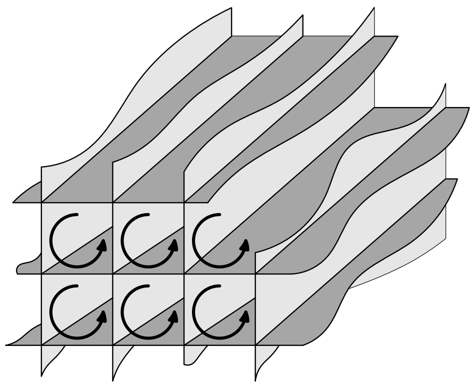

Figure 5: The picture of a 2-form.

Forms of higher degree The basic forms with a degree greater than 1 are the wedge products of the coordinate differentials $\{\boldsymbol{dx}^i\}$. One can imagine a basic 2-form as a set of orientated tubes "field lines", as it is shown in figure 5. It is important to note that the shape of the tubes has no meaning, just their density and their orientation is of significance: One can always write

$$\boldsymbol{dx}^1 \wedge \boldsymbol{dx}^2 \equiv (2 \cdot \boldsymbol{dx}^1) \wedge (\frac{1}{2} \cdot \boldsymbol{dx}^2).$$

An important example is the already mentioned Faraday tensor $\boldsymbol{F}$ and its dual $*\boldsymbol{F}$. They contain all information on the electromagnetic field, and the corresponding picture is that of orientated field lines or field tubes. Now we can give an interpretation of Maxwell's equations:

- $\boldsymbol{dF} = 0$: The tubes of the Faraday tensor never end.
- $\boldsymbol{d} * \boldsymbol{F} = 4\pi * \boldsymbol{j}$: The tubes of the Maxwell tensor end in charges or currents.

A 3-form, for $n \geq 3$, can be imagined as a stack of orientated cubes. A physical application is the notion of the dual charge density $*\boldsymbol{j}$ in electrodynamics. The integration of this 3-form over a 3-dimensional hypersurface gives the total electric charge in this volume. Another important n-form on a manifold is the volume form $\boldsymbol{\eta}$. In terms of the determinant $\sqrt{|g|}$ it reads

$$\boldsymbol{\eta} = \sqrt{|g|} \cdot \boldsymbol{\epsilon} = \sqrt{|g|} \cdot \boldsymbol{dx}^1 \wedge \boldsymbol{dx}^2 \wedge ... \wedge \boldsymbol{dx}^n .$$

It is used to calculate volumes. There are two reasons: From the Riemann integral one knows that the 'volume element' $dx^1...dx^n$ is not invariant under change of coordinates. A prefactor occurs which is compensated by the prefactor of $\sqrt{|g|}$. (The volume element is a scalar density of weight -1 while $\sqrt{|g|}$ has the weight $+1$.) The other reason is that in the case of embedded manifolds one can intuitively deduce a volume element which turns out to be proportional to $\sqrt{|g|}$.
As instructive example of the calculus of forms we refer to page 72.

Frobenius' theorem

We give some information on the background of Frobenius' theorem. It follows the argumentation of [Wald(1984)], Appendix B.3.

Introduction An issue which is somewhat related to forms - as it deals with hypersurfaces - is the question of the integrability of surfaces. What is meant by this term will first be explained by an introductory example: Consider a manifold $\mathcal{M}$ with metric g and a vector field ξ defined on it. At a given point $p \in \mathcal{M}$, ξ defines a direction in the tangent space $T_p\mathcal{M}$ and thus a $(n-1)$-dimensional subspace $W_p \subset T_p\mathcal{M}$ which is orthogonal to ξ in p. One may wish to know whether ξ is orthogonal to a family of hypersurfaces. This is the case if there is a 1-parameter family of curves $x^a = x^a(s)$ whose tangent vector field is given by ξ. Then the vector field is called *integrable*. Thus one has to solve the ordinary differential equation

$$\frac{dx^a}{ds} = \xi^a$$

for the vector field x^a. This is an ordinary differential equation and the existence of a solution is guaranteed by the theorem of Picard-Lindelöf for a Lipschitz-continous ξ.

The theorem and its proof The aim of the Frobenius theorem is to make statements on the existence of integral submanifolds for a system of *more than one* orthogonal vector fields (or 1-forms). Let their number be m.
The question is wether the family W of all $(n-m)$-dimensional W_p is integrable, this means, wether there is an embedded submanifold $\mathcal{S} \subset \mathcal{M}$ with $\forall p \in \mathcal{S} : T_p\mathcal{S} = W_p$. This can imply that a manifold can locally be written as a product manifold, consider e.g. the metric of a Kerr Black Hole.

Let us first assume that there exists such a submanifold. Then we locally could find coordinate vector fields $x_1, ...x_m$ on $\mathcal{M}$ which for each p in an open neighborhood of a p_0 span the subspace W_p and the commutators of all x_i lie in the span of the x_i: $[x_i, x_j] = \sum_k C_{ijk} x_k$.

Any two vector fields y and z can be written as linear combinations of the x_i, and for the commutator $[y,z]$ one finds:

$$[y,z] = [\sum_\mu a_\mu x_\mu, \sum_\nu b_\nu x_\nu] = \sum_{\mu\nu} a_\mu b_\nu \sum_\rho C_{\mu\nu\rho} x_\rho = \sum_\rho (\sum_{\mu\nu} a_\mu b_\nu C_{\mu\nu\rho}) x_\rho \in W.$$

The property of W, that for all vector fields the commutator of two of them lies again in W, is called *involutivity.* It is a neccessary property of W to possess an integral submanifold. This condition is also sufficient, which is the statement of Frobenius' theorem:

Frobenius theorem, vector form: A smooth specification[3] W of m-dimensional subspaces in each $p \in \mathcal{M}$ possesses an integral submanifold if and only if W is involute, that is, if and only if $\forall y, z \in W : [y,z] \in W$.

Now we want to reformulate the theorem in terms of forms.
For fixed p we consider the annihilator W_p^* of W_p in $T_p^*\mathcal{M}$ with elements $\boldsymbol{\omega}$ such that

$$(\boldsymbol{\omega}|X) = \boldsymbol{\omega}_a X^a = 0$$

for all $X^a \in W_p$. Each W_p defines such a $(n-m)$-dimensional annihilator and vice versa. Now we reformulate the above question:
When does a smooth specification W^* of $(n-m)$-dimensional subspaces of 1-forms have the property that their annihilators W admit integral submanifolds?
Frobenius' theorem states that W is integrable if and only if $[y,z] \in W$. Then,

$$\boldsymbol{\omega}([y,z]) = \boldsymbol{\omega}_\rho (\sum_{\mu\nu} a_\mu b_\nu C_{\mu\nu\rho}) x_\rho = 0.$$

As y and z are linear combinations of the x^i, one has $\boldsymbol{\omega}_a y^a = 0$; $\boldsymbol{\omega}_a z^a = 0$. On a manifold with vanishing torsion, one can express the commutator of two arbitrary vector fields X and Y by *any* derivative operator:

$$\begin{aligned} X = X^a \partial_a \quad \Longrightarrow \quad & [X,Y]^a = (X^b \partial_b Y^c \partial_c - Y^b \partial_b X^c \partial_c)^a = \\ & ((X^b \partial_b Y^c - Y^b \partial_b X^c)\partial_c)^a = X^b \partial_b Y^a - Y^b \partial_b X^a. \end{aligned} \tag{A-15}$$

Now we can calculate the consequences for $\boldsymbol{\omega}^i$:

$$\begin{aligned} 0 &= \boldsymbol{\omega}_a [y,z]^a \\ &= \boldsymbol{\omega}_a (y^b \partial_b z^a - z^b \partial_b y^a) \\ &\overset{\partial_b \boldsymbol{\omega}_a y^a = 0}{=} -z^a y^b \partial^b \boldsymbol{\omega}_a + z^b y^a \partial^b \boldsymbol{\omega}_a \\ &= 2 y^a z^b \partial_{[b} \boldsymbol{\omega}_{a]}. \end{aligned} \tag{A-16}$$

The terms $\partial_{[b}\boldsymbol{\omega}_{a]}$ are the components of the the 2-form $\boldsymbol{d\omega}$. It follows

$$\boldsymbol{d\omega}(y,z) = 0.$$

[3]'smooth specification': The subspaces W_p at each $p \in \mathcal{M}$ vary smoothly with p, that means that for any $p \in \mathcal{M}$ there is an open neighbourhood U of p such that for all $q \in U$ the W_q are spanned by C^∞-vector fields.

y and z are elements of W, while $\boldsymbol{\omega}$ is an element of the annihilator of W. This is possible if and only if one can write $\boldsymbol{d\omega}$ as

$$\boldsymbol{d\omega} = \sum_\alpha \boldsymbol{\mu}^\alpha \wedge \boldsymbol{\nu}^\alpha,$$

where $\boldsymbol{\mu}^\alpha \in T^*$ and $\boldsymbol{\nu}^\alpha$ is an arbitrary 1-form. This is the
Dual formulation of Frobenius' theorem: Let T^* be a smooth specification of a $(n-m)$-dimensional subspace of one-forms. Then the annihilator $W \subset T$ admits an integral submanifold if and only if

$$\forall \boldsymbol{\omega} \in T^* : \quad \boldsymbol{d\omega} = \sum_\alpha \boldsymbol{\mu}^\alpha \wedge \boldsymbol{\nu}^\alpha, \qquad \boldsymbol{\mu}^\alpha \in T^* \text{ and } \boldsymbol{\nu}^\alpha \text{arbitrary.}$$

Applications As a first example we consider the question wether a given vector field ξ is orthogonal to a family of hypersurfaces. At $p \in \mathcal{M}$, the tangent space $T_p\mathcal{M}$ and its dual $T_p^*\mathcal{M}$ are one dimensional. The form $\boldsymbol{\xi}_a = g_{ab}\xi^a$ is a basis of T^*. In one dimension the number of forms $\boldsymbol{\mu}^\alpha$ reduces to one and can be set equal to $\boldsymbol{\xi}$. We have $\boldsymbol{d\xi} = \boldsymbol{\xi} \wedge \boldsymbol{\nu}$. One can eliminate $\boldsymbol{\nu}$ by showing that this condition is equivalent to

$$\boldsymbol{\xi} \wedge \boldsymbol{d\xi} = 0$$

or the more familiar expression in components

$$\xi_{[a}\partial_b\xi_{c]} = 0.$$

Here the dual version of the theorem has been used.

Another example occurs in a stationary and axisymmetric spactime $\mathcal{M}$ which is asymptotically flat. There are two Killing vectors, a timelike one ξ and a spacelike one η with closed orbits. Of each of them one can determine the integral curves separately. As a consequence of asymptotic flatness both vectors commute, $[\xi, \eta] = 0$. At each point p of $\mathcal{M}$, consider the two-dimensional subspace W of $T_p\mathcal{M}$ which is spanned by ξ and η. As ξ and η commute, we have $\forall y, z \in W : \quad [y, z] \in W$. From the vector version of Frobenius's theorem we can conclude that
there is an embedded submanifold $S \subset \mathcal{M}$ whose tangential space is spanned by ξ and η at each point. Further on, this allows us to foliate the spacetime locally:

$$(\mathcal{M}, g) = (\bigcirc, \sigma) \times (\perp, \tau) \tag{A-17}$$

Here, the two-dimensional manifold $\bigcirc$ is called *orbit manifold* and is pseudo-Riemannian, that means that its metric has signature 1. It's tangent bundle is spanned by ξ and η. One can further parametrize it as $\bigcirc = \mathbb{R} \times SO(2)$ which reflects the time translation symmetry and the invariance under rotations. The symmetry adapted coordinates t and ϕ refer to the Killing vectors

$$\xi \stackrel{*}{=} \frac{\partial}{\partial t}, \qquad \eta \stackrel{*}{=} \frac{\partial}{\partial \phi}.$$

$\perp$ is called the *orthogonal manifold.* It is two-dimensional and Riemannian.
One can therefore split the calculation of derivatives and other operations into a part on $\bigcirc$ and a one on $\perp$.
Because of stationarity and axisymmetry all functions on $\mathcal{M}$ do not depend on $\bigcirc$.
The determinant $|g|$ is a product $|g| = |\sigma| \cdot |\tau|$ which also only depends on $\perp$.

Appendix A

The multiplication law for matrix determinants

As an application of the calculus of differential forms, one can give a short proof of the multiplication property of matrix determinants.
As manifold we choose $\mathbb{R}^n$ with cartesian coordinates $(x^{(1)}, x^{(2)}, ..., x^{(n)})$. A matrix B is then considered as a linear operator which transforms to new coordinates: $x'^i = B^i_j x^j$. Its determinant can be written as $|B| = B^1_{a_1} \cdot B^2_{a_2} \cdot ... \cdot B^n_{a_n} \cdot \epsilon^{a_1 a_2 ... a_n}$.
Consider two matrices A, B which transform the unprimed into primed and further to double-primed coordinates:

$$(x) \xrightarrow{B} (x') \xrightarrow{A} (x'').$$

Now one can define special $(n-1)$-forms in each of these systems

$$\begin{aligned} \boldsymbol{x''} &:= x''^{(1)} \cdot \boldsymbol{dx''}^{(2)} \wedge \boldsymbol{dx''}^{(3)} \wedge ... \wedge \boldsymbol{dx''}^{(n)} \\ \boldsymbol{x'} &:= x'^{(1)} \cdot \boldsymbol{dx'}^{(2)} \wedge \boldsymbol{dx'}^{(3)} \wedge ... \wedge \boldsymbol{dx'}^{(n)} \\ \boldsymbol{x} &:= x^{(1)} \cdot \boldsymbol{dx}^{(2)} \wedge \boldsymbol{dx}^{(3)} \wedge ... \wedge \boldsymbol{dx}^{(n)}. \end{aligned}$$

$\boldsymbol{x''}$ can be expressed in primed coordinates

$$\boldsymbol{x''} = x''^1 \cdot \boldsymbol{dx''}^{(2)} \wedge \boldsymbol{dx''}^{(3)} \wedge ... \wedge \boldsymbol{dx''}^{(n)} = A^1_{a_1} \cdot x'^{a_1} \cdot A^2_{a_2} \cdot \boldsymbol{dx'}^{(a_2)} \wedge ... \wedge A^n_{a_n} \boldsymbol{dx'}^{(a_n)}.$$

Analogously for $\boldsymbol{x'}$. Note that both the coordinates and the basic forms (the 'coordinate differentials') are contravariant objects. Now one calculates the exterior derivative $\boldsymbol{dx''}$:

$$\begin{aligned} \boldsymbol{dx''} &= \boldsymbol{dx''}^{(1)} \wedge \boldsymbol{dx''}^{(2)} \wedge ... \wedge \boldsymbol{dx''}^{(n)} \\ &= A^1_{a_1} \cdot A^2_{a_2} \cdot ... \cdot A^n_{a_n} \cdot \boldsymbol{dx'}^{(a_1)} \wedge ... \wedge \boldsymbol{dx'}^{(a_n)} \\ &= A^1_{a_1} \cdot A^2_{a_2} \cdot ... \cdot A^n_{a_n} \epsilon^{a_1 a_2 ... a_n} \cdot \boldsymbol{dx'}^{(1)} \wedge ... \wedge \boldsymbol{dx'}^{(n)} \\ &= |A| \cdot \boldsymbol{dx'}. \end{aligned}$$

By means of this construction one obtains a formula where $|A|$ occurs in a natural way. Analogously, we get $\boldsymbol{dx'} = |B| \cdot \boldsymbol{dx}$. One can consider the changes in coordinates $(x) \longrightarrow (x'')$ as two successive transforms and also as a single one, carried out by $A \cdot B$: $x''^i = A^i_j B^j_k x^k = (A \cdot B)^i_k x^k$. This reflects the associative law of matrix multiplication.
Finally, we have

$$\begin{aligned} \boldsymbol{dx''} &= |A \cdot B| \cdot \boldsymbol{dx} \\ \boldsymbol{dx''} &= |A| \cdot \boldsymbol{dx'} \\ &= |A| \cdot |B| \cdot \boldsymbol{dx}. \end{aligned}$$

and finally

$$|AB| = |A||B|.$$

This demonstrates the effectivity of some calculations with differential forms.

Appendix B

Stationary and axisymmetric spacetimes in Weyl coordinates

For purposes of completeness and for review we compile some notation of the s&a line element in Weyl coordinates and some metric functions of the Kerr Black Hole.

The line element in Weyl coordinates

$$ds^2 = e^{-2U}(e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\phi^2) - e^{2U}(dt + ad\phi)^2 \tag{B-1}$$

It depends on three metric functions U, a, k which depend only on (ρ, ζ). The metric tensor is

$$g_{ab} = \begin{pmatrix} e^{-2U}e^{2k} & 0 & 0 & 0 \\ 0 & e^{-2U}e^{2k} & 0 & 0 \\ 0 & 0 & \rho^2 e^{-2U} - a^2 e^{2U} & -ae^{2U} \\ 0 & 0 & -ae^{2U} & -e^{2U} \end{pmatrix}_{ab} . \tag{B-2}$$

The inverse metric tensor reads:

$$g^{ab} = \begin{pmatrix} e^{2U}e^{-2k} & 0 & 0 & 0 \\ 0 & e^{2U}e^{-2k} & 0 & 0 \\ 0 & 0 & \frac{e^{2U}}{\rho^2} & -\frac{ae^{2U}}{\rho^2} \\ 0 & 0 & -\frac{ae^{2U}}{\rho^2} & \frac{a^2e^{2U}}{\rho^2} - e^{-2U} \end{pmatrix}^{ab} . \tag{B-3}$$

The Ernst equation

In this special coordinate system Einstein's equations reduce to the Ernst equation for the Ernst function (or 'Ernst potential') f:

$$\Re f(f,_{\rho\rho} + f,_{\zeta\zeta} + \frac{1}{\rho} f,_{\rho}) = f,_{\rho}^2 + f,_{\zeta}^2 , \tag{B-4}$$

or independently from the choice of coordinates

$$\Re f \triangle^{(g)} f = (\boldsymbol{d}f|\boldsymbol{d}f)^{\perp}, \tag{B-5}$$

where

$$f(\rho, \zeta) := e^{2U} + ib.$$

b replaces a via the system

$$a_{,\rho} = \rho e^{-4U} b_{,\zeta}\,; \qquad\qquad a_{,\zeta} = -\rho e^{-4U} b_{,\rho}\,, \tag{B-6}$$

and k can be calculated by

$$k_{,z} = \frac{\rho}{2} e^{-4U} (f_{,z}\, f_{,\bar{z}}\,). \tag{B-7}$$

The Kerr metric

The Kerr solution depends on two parameters. Here we choose the mass M and the angular momentum J of the Black Hole. (One could also take the angular velocity of the horizon and the expansion of the horizon on the ζ-axis, for example.) For the rotating Black Hole spacetime one often uses the *Boyer-Lindquist*-coordinates (r, θ) instead of (ρ, ζ) on the orthogonal manifold. They are related by

$$\rho = \sqrt{r^2 - 2Mr + \frac{J^2}{M^2}} \cdot \sin\theta; \qquad\qquad \zeta = (r - M)\cos\theta. \tag{B-8}$$

The metric funtions now read

$$e^{2U} = \frac{r^2 - 2Mr + \frac{J^2}{M^2}\cos\theta^2}{r^2 + \frac{J^2}{M^2}\cos\theta^2}, \tag{B-9}$$

$$a = \frac{2Jr\sin\theta^2}{r^2 - 2Mr + \frac{J^2}{M^2}\cos\theta^2}, \tag{B-10}$$

$$b = \frac{-2J\cos\theta}{r^2 + \frac{J^2}{M^2}\cos\theta^2}, \tag{B-11}$$

$$e^{2k} = \frac{r^2 - 2Mr + \frac{J^2}{M^2}\cos\theta^2}{r^2 - 2Mr + \frac{J^2}{M^2}\cos\theta^2 + M^2\sin\theta^2}. \tag{B-12}$$

In Weyl coordinates however the Kerr metric looks more complicated. With the abbreviation

$$r_\pm := \sqrt{\rho^2 + \left(\zeta \pm \sqrt{M^2 - \frac{J^2}{M^2}}\right)^2},$$

the Ernst potential reads

$$f(\rho, \zeta) = 1 - \frac{4M}{r_+ + r_- + 2M - i\frac{J(r_+ - r_-)}{\sqrt{M^4 - J^2}}}. \tag{B-13}$$

The Ernst potential in a rotating frame

Under a one-parameter coordinate transformation

$$\rho' := \rho; \qquad \zeta' := \zeta; \qquad t' := t; \qquad \phi' := \phi - \omega t$$

the metric remains form-invariant. The tilded, new metric functions are related to the untilded by

$$\begin{aligned}
e^{2\tilde{U}} &= e^{2U} + 2a \cdot e^{2U}\omega + (e^{2U} \cdot a^2 - \rho^2 \cdot e^{-4U})\omega^2, \\
\tilde{a} &= (a \cdot e^{2U} + a^2 \cdot e^{2U}\omega - \rho^2 \cdot e^{-2U}\omega)e^{-\tilde{2}U}, \\
e^{2\tilde{k}} &= e^{2(k-U)} \cdot e^{2\tilde{U}} = e^{2k}(1 + 2a \cdot \omega + (a^2 - \rho^2 \cdot e^{-4U})\omega^2).
\end{aligned}$$

The *corotating Ernst potential* of the Kerr metric can be determined to

$$\begin{aligned}
f &= e^{2U} + (2a \cdot e^{2U})\omega + (e^{2U} \cdot a^2 - \rho^2 \cdot e^{-4U})\omega^2 \\
&+ i\Bigg(\frac{-2J\cos\theta}{r^2 + \frac{J^2}{M^2}\cos\theta^2} + \Bigg(- 2\cos\theta(r-M) + 4M\cos\theta\frac{r^2 + \frac{J^2}{M^2}}{r^2 + \frac{J^2}{M^2}\cos\theta^2}\Bigg)\omega \\
&\quad + \Bigg(J(2\cos\theta^3 - 6\cos\theta) - \frac{J\frac{J^2}{M^2}\sin\theta^4\cos\theta}{r^2 + \frac{J^2}{M^2}\cos\theta^2}\Bigg)\omega^2\Bigg).
\end{aligned}$$

The surface gravity of a Kerr Black Hole

The surface gravity κ of a stationarily rotating Black Hole reads

$$\kappa_{\mathrm{Kerr-b.h.}} = \frac{\sqrt{M^2 - \frac{J^2}{M^2}}}{2M\Big(M + \sqrt{M^2 - \frac{J^2}{M^2}}\Big)}. \tag{B-14}$$

For an extreme Kerr Black Hole the surface gravity vanishes

$$\kappa_{\mathrm{eKerr-b.h.}} = 0. \tag{B-15}$$

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Selbständigkeitserklärung

Mit meiner Unterschrift versichere ich, daß ich diese Arbeit selbständig und nur unter Verwendung der angegebenen Quellen angefertigt habe.

Jena, den 23. Februar 2007,

Seitens des Verfassers bestehen keine Einwände, die vorliegende Diplomarbeit für die öffentliche Nutzung in der Thüringer Universitäts- und Landesbibliothek zur Verfügung zu stellen.

Jena, den 23. Februar 2007,

Danksagung

Mein Dank gilt Herrn Professor Meinel für die Betreuung der Arbeit und die zahlreichen Gespräche, die stets in freundlicher und entspannter Atmosphäre verliefen.
Weiterhin möchte ich mich bei den Betreuern der Studienstiftung des deutschen Volkes für die kritische Begleitung meines Studiums danken. Das von der Stiftung über fünf Jahre gewährte Stipendium hat mir das Studium ermöglicht.
Meine Familie hat mich während dieser Zeit ebenfalls unterstützt und ermutigt.
Dr. David Petroff hat große Teile der Arbeit Korrektur gelesen. Ihm möchte ich an dieser Stelle herzlich danken.